Problem 10

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Problem 10 (1.6): Prove the fundamental theorem concerning branching processes by utilizing the theory of absorption probabilities.

Let us denote the number of individuals in the n-th generation by $Z^{(n)}$. This is a random variable and can be defined by a Markov Chain. The Transition Probability p_{jk} is defined as the probability that $Z^{(n+1)} = k$ given that $Z^{(n)} = j$.We define the generating function as **(Bharucha-Reid, 1.28)**:

$$F(x) = \sum_{k=0}^{\infty} p_k x^k$$

Due to the assumption of statistical independence we have **(Bharucha-Reid, 1.29)**:

$$p_{jk} = \text{the coefficient of } x^k \text{ in } F^j(x)$$

From this we can say that if the original population size is r, then the generating function of the population size $Z^{(n)}$ in the n-th generation is given by $F_n^r(x)$, where **(Bharucha-Reid, 1.30)**:

$$F_1(x) = F(x),\ F_2(x) = F(F(x)),\ \ldots\ ,\ F_{n+1}(x) = F(F_n(x))$$

Now we consider the absorption probability α_r that the population size will eventually be reduced to 0 if the initial population is r. This can only be possible if the absorption state has a non-zero probability. Then the absorption probability must be a solution of the infinite system of equations **(10.1)**:

$$\alpha_r = \sum_{v=1}^{\infty} p_{rv} \alpha_v + p_{r0}$$

Now we might see that we can get a solution of the form $\alpha_r = \lambda^r$. With the R.H.S. also expressed as $F^r(x)$, and therefore (10.1) implies that $\lambda^r = F^r(\lambda)$. We write this as **(10.2)**:

$$\lambda = F(\lambda)$$

If λ is a solution of (10.2), then $\alpha_r = \lambda^r$ is a solution of (10.1). We know this from **(Bharucha-Reid, 1.4.C, pg.25)** that $\lambda = 1$ is always a solution of (10.2), but if $F'(1) > 1$ there exists another root $\lambda < 1$. Then the previous solution will not be unique (look at Bharucha-Reid, figure 1.2 on pg. 26). In this case the smallest solution will determine the absorption probability.