Problem 11

Osama K Mahmood

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Problem 11 (2.10): Consider a birth-and-death process with parameters λ_x and μ_x. Let T_n denote the time required for the random variable to increase from n to $n+1$ and let $T_n^* = E\{T_n\}$. T_n^* is the conditional expected time, conditioned upon non-absorption or extinction. Show that $T_n^* = \frac{1}{\lambda_n} + \frac{\mu_n}{\lambda_n} T_{n-1}^*$

The probability density function for the time t elapsing until the occurrence of the first event population size has reached n is defined by **(11.1)**:

$$f(t) = (\lambda_n + \mu_n)\exp[-(\lambda_n + \mu_n)t]$$

This event has a probability $\frac{\lambda_n}{\lambda_n+\mu_n}$ of being a birth and in this case the population will increase from n to $n+1$. It has a probability $\frac{\mu_n}{\lambda_n+\mu_n}$ of being a death and in this case the population will reduce from n to $n-1$. Now we must have another passage from $n-1$ to n and then one from passage from n to $n+1$. So we have the relation **(11.2)**:

$$T_n^* = \frac{\lambda_n}{\lambda_n+\mu_n} \frac{1}{\lambda_n+\mu_n} + \frac{\mu_n}{\lambda_n+\mu_n} \left(\frac{1}{\lambda_n+\mu_n} + T_{n-1}^* + T_n^*\right)$$

This solves to the desired equation that **(11.3)**:

$$T_n^* = \frac{1}{\lambda_n} + \frac{\mu_n}{\lambda_n} T_{n-1}^*$$