Problem 12

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Problem 12 (2.13): The Coefficient of Variation of the random variable $X(t)$ is defined as:

$$\nu[X(t)] = \frac{D[X(t)]}{E[X(t)]}$$

the ration of standard deviation and mean of $X(t)$. Determine the Coefficient of Variation and its Asymptotic Behavior for (a) Poisson Process, (b) Simple Birth Process and (c) Birth-and-Death Process.

Following is the derivation of the Coefficient of Variation for

(a) Poisson Process
We know that (Bharucha-Reid, 2.94):

$$E[X(t)] = \lambda t$$

And that (Bharucha-Reid, 2.95):

$$D^2[X(t)] = \lambda t \Rightarrow D[X(t)] = \sqrt{\lambda t}$$

Therefore the Coefficient of Variation is computed as:

$$\nu[X(t)] = \frac{D[X(t)]}{E[X(t)]} = \frac{\sqrt{\lambda t}}{\lambda t} = \frac{1}{\sqrt{\lambda t}}$$

As $t \to \infty$:

$$\nu[X(t)] \to 0$$

(b) Simple Birth Process
We know that (Bharucha-Reid, 2.110):

$$E[X(t)] = e^{\lambda t}$$

And that (Bharucha-Reid, 2.111):

$$D^2[X(t)] = e^{\lambda t}(e^{\lambda t} - 1) \Rightarrow D[X(t)] = e^{(\lambda t)/2}\sqrt{e^{\lambda t} - 1}$$

Therefore the Coefficient of Variation is computed as:

$$\nu[X(t)] = \frac{D[X(t)]}{E[X(t)]} = \frac{e^{(\lambda t)/2}\sqrt{e^{\lambda t}-1}}{e^{\lambda t}} = \sqrt{1 - e^{-\lambda t}}$$

As $t \to \infty$:

$$\nu[X(t)] \to 1$$

(c) Birth-and-Death Process
We know that (Bharucha-Reid, 2.154):

$$E[X(t)] = e^{(\lambda-\mu)t}$$

And that (Bharucha-Reid, 2.155):

$$D^2[X(t)] = \frac{\lambda+\mu}{\lambda-\mu} e^{(\lambda-\mu)t}(e^{(\lambda-\mu)t}-1) \Rightarrow D[X(t)] = \sqrt{\frac{\lambda+\mu}{\lambda-\mu}} e^{((\lambda-\mu)t)/2} \sqrt{e^{(\lambda-\mu)t}-1}$$

Therefore the Coefficient of Variation is computed as:

$$\nu[X(t)] = \frac{D[X(t)]}{E[X(t)]} = \frac{\sqrt{\frac{\lambda+\mu}{\lambda-\mu}} e^{((\lambda-\mu)t)/2} \sqrt{e^{(\lambda-\mu)t}-1}}{e^{(\lambda-\mu)t}} = \sqrt{\frac{\lambda+\mu}{\lambda-\mu}(1-e^{-(\lambda-\mu)t})}$$

As $t \to \infty$:

$$\nu[X(t)] \to \sqrt{\frac{\lambda+\mu}{\lambda-\mu}}$$