Problem 1.1 (Soln)

We will prove that $tr(AB - BA) \neq tr(I_n)$

$tr(I_n) = n$ simply.

$$tr(AB - BA) = tr(AB) - tr(BA).$$

$$AB = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & & & \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & & & & \\ b_{31} & & & & \\ \vdots & & & & \\ b_{n1} & & & & \end{pmatrix}$$

We see that $(ab)_{11} = \sum_{i=1}^{n} a_{1i} b_{i1}$

$$(ab)_{22} = \sum_{i=1}^{n} a_{2i} b_{i2}$$

$$\vdots$$

$$(ab)_{nn} = \sum_{i=1}^{n} a_{ni} b_{in}$$

$$\Rightarrow tr(AB) = \sum_{i=1}^{n} (ab)_{ii}$$

$$tr(AB) = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ji} b_{ij}$$

Similarly $tr(BA) = \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ji} a_{ij}$

If there is an entry a_{mp} then in the $tr(AB)$ it will appear as $a_{mp} b_{pm}$ $\forall p, m \leq n$.

Hence it is clear by inspection that $tr(AB) = tr(BA)$

$$\Rightarrow tr(AB - BA) = 0.$$

$\therefore$ Hence we cannot have such matrices.

Problem 1.2 (Soln)

$$A^2 = rB^2 \quad \text{--- (1)}$$

$$A^4 = rA^2B^2 \quad \text{--- (1a)}$$

$$A^4 = rB^2A^2 \quad \text{--- (1b)} \Longrightarrow A^2B^2 = B^2A^2 = r'A^4 \quad (\text{From (1)})$$

$$rB^2A = rAB^2 = A^3 \quad (\text{Again from (1)})$$

$$pAB + qBA = I_n \quad \text{--- (2)}$$

$$\Rightarrow pBAB + qB^2A = B \quad \text{--- (3)}$$

$$\Rightarrow pAB^2 + qBAB = B \quad \text{--- (4)}$$

Working

$$pBAB + qAB^2 = B$$

$$pAB^2 + qBAB = pBAB + qAB^2$$

$$(q-p)BAB = (q-p)AB^2$$

$$= (q-p)B^2A$$

Combining (3) and (4) and using the fact that $B^2A = AB^2$, we get

$$BAB = AB^2 = B^2A$$

$$p(BAB) + qB^2A = B$$

$$\Rightarrow (p+q)B^2A = B$$

$$(p+q)B^2A^2 = BA \Longrightarrow (p+q).AB^2 = B$$

$$(p+q)A^2B^2 = AB$$

From (1a) and (1b) we know (A^2) and (B^2) commute.

So A and B also commute. Contradiction!

Problem 2.1 (Soln)

If $M = A + iB \Longrightarrow \overline{M^t} = \overline{A^t} - i\overline{B^t} = A - iB$

Since we should satisfy the above condition we can pick,

$$A = \frac{1}{2}(M + \overline{M^t}) \quad B = \frac{1}{2i}(M - \overline{M^t}),$$

which are both Hermitian. $\square$

Examples of Hermitian Matrices,

$$\begin{pmatrix} 2 & -i \\ i & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 1+i & 2i \\ 1-i & 5 & -3 \\ -2i & -3 & 0 \end{pmatrix}.$$

Problem 3.1 (Soln)

Consider the matrix

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

This matrix has the property that

$$M^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

We have $\det(M^n) = F_{n+1}F_{n-1} - F_n^2 = (\det M)^n = (-1)^n$ $\square$

Remember that $\det(AB) = \det(A)\det(B)$

or $\det(A_1 A_2 \ldots A_n) = \det(A_1)\det(A_2)\ldots\det(A_n)$.

Problem 3.2 (Soln)

Lemma: $\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - CB)$,

given that $AC = CA$.

Proof: The Rule of Laplace says that for

$$Z = \begin{pmatrix} A & 0 \\ C & I_n \end{pmatrix}$$

$$\det(Z) = \det(A)\det(I_n) = \det(A)$$

or if $Z = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \Rightarrow \det(Z) = \det(A)\det(B)$

Observe that, $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & I_n \end{pmatrix}\begin{pmatrix} I_n & A^{-1}B \\ 0 & D - CA^{-1}B \end{pmatrix}$

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\det(I_n)\det(I_n)\det(D - CA^{-1}B)$$

$$= \det(AD - ACA^{-1}B) = \det(AD - CB) \quad //$$

Coming back,

$$\det(I_n - YX) = \det\begin{pmatrix} I_n & X \\ Y & I_n \end{pmatrix} = (-1)^n \det\begin{pmatrix} Y & I_n \\ I_n & X \end{pmatrix}$$

$$= (-1)^{2n} \det\begin{pmatrix} I_n & Y \\ X & I_n \end{pmatrix}$$

$$= \det(I_n - XY) \quad \square$$

Problem 4.1 (Soln)

Observe that the inverse of a 2×2 matrix (A) with integer entries is a matrix with integer entries if and only if $\det(A) = \pm 1$.

Because we know that $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Let us consider a polynomial $P(x) \in \mathbb{Z}(x)$, $P(x) = \det(A + xB)$

The problem tells us that $P(0), P(1), P(2), P(3), P(4) \in \{-1, 1\}$.

By the pigeonhole principle three of these must be equal, and because $P(x)$ has degree at most 2, it must be constant.

Therefore $\det(A + xB) = \pm 1$.

The fact is true for all $x \in \mathbb{Z}^+$ and so is true for $x = 5$ $\square$

Problem 4.2 (Soln)

Recall the identity,

$$(x-1)(1+x+x^2+\dots+x^k) = x^{k+1}-1$$

$$1+x+x^2+\dots+x^k = \frac{x^{k+1}-1}{x-1}$$

By differentiation we get,

$$1+2x+3x^2+\dots+kx^{k-1} = \frac{kx^{k+1}-(k+1)x^k+1}{(x-1)^2}$$

Substituting A for x we get,

$$(A-I_n)^2(I_n+2A+3A^2+\dots+kA^{k-1})$$

$$= kA^{k+1}-(k+1)A^k+I_n = I_n$$

Multiplying both sides by $(A-I_n)^{-1}$ we get

$$(A-I_n)^{-1} = (A-I_n)(I_n+2A+3A^2+\dots+kA^{k-1}) \quad \square$$