Problem 4

Osama K Mahmood

May 2016

Problem 4 (1.4): Calculate the higher moments and cumulants of X_n, where X_n is a simple branching process.

The Moment Generating Function of the random variable X is given by **(1.28)**:
$M(t) = E[e^{tX}]$ setting $s = e^t$, therefore:
$M(t) = E[1 + Xt + \frac{X^2}{2!}t^2 + \frac{X^3}{3!}t^3 + ...]$

Just as done in **Bharucha-Reid, pg. 20** we will compute the third cumulant in terms of the generating functions. As compiled in **(1.36)**, we get the following result **(4.1)**:

$i_3 = D^3(X_1) = F'''(1) - F''(1) - 3F'(1) - 3F'(1)F''(1) - 3[F'(1)]^2 + 2[F'(1)]^3$

And as in **(1.37)**, we get the following expression **(4.2)**:

$$D^3(X_n) = F_n'''(1) - F_n''(1) - 3F_n'(1) - 3F_n'(1)F_n''(1) - 3[F_n'(1)]^2 + 2[F_n'(1)]^3$$

Now we will proceed as we proceeded in equations **(1.38)** and **(1.39)**. By differentiating **(1.38)** we get **(4.3)**:

$$F_{n+1}'''(1) = F_n'''(1)[F'(1)]^3 + 3F_n''(1)F'(1)F''(1) + F_n'(1)F'''(1)$$

And by differentiating **(1.39)** we get **(4.4)**:

$$F_{n+1}'''(1) = F'''[F_n(1)][F_n'(1)]^3 + 3F''[F_n(1)]F_n'(1)F_n''(1) + F'[F_n(1)]F_n'''(1)$$

Using **(4.1)** we also find the following explicit expression for $F'''(1)$ **(4.5)**:

$$F'''(1) = i_3 + \sigma^2 + m^2 + 2m + 3m\sigma^2 + m^3$$

We equate **(4.3)** and **(4.4)** and we get the following expression **(4.6)**:

$$F_n'''(1) = (i_3 + \sigma^2 + m^2 + 2m + 3m\sigma^2 + m^3)\frac{m^{n-1}(m^{2n}-1)}{m^2-1}$$

The expression for $D^3(X_n)$ follows by putting the right expressions in **(4.2)**

$$D^3(X_n) =$$
$$F'''(1)\frac{m^{n-1}(m^{2n}-1)}{m^2-1} - (3m+1)\sigma^2 m^n \frac{m^n-1}{m^2-m} - m^n(m^n-1) - 3m^n - 3m^{2n} + 2m^{3n}$$

.