Problem 5

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Problem 5 (2.2): Let $X(t), t \geq 0$ be a time-homogeneous stochastic process with independent increments and let:

$$Y_i(t) = P([X(t) - X(0)] = i)$$

satisfy the conditions:

$$\lim_{t \to 0} \frac{Y_1(t)}{t} = \lambda > 0; \lim_{t \to 0} \frac{1 - Y_0(t) - Y_1(t)}{t} = 0 \tag{1}$$

Show that:

$$Y_i(t) = \frac{(\lambda t)^i}{i!} e^{-\lambda t} \tag{2}$$

for i=0, 1, ...

By adding up the two equations given in **(7)** we get:

$$\lim_{t \to 0} \frac{1 - Y_0(t)}{t} = \lambda \tag{3}$$

Equating this result with the first equation in **(7)** we get:

$$Y_1(t) + Y_0(t) = 1 \tag{4}$$

From equation **(9)** we get:

$$\lim_{t \to 0}(1 - Y_0(t)) = \lambda t \to \lim_{t \to 0} Y_0(t) = 1 - \lambda t \tag{5}$$

By substituting **(10)** in **(11)** we get that:

$$\lim_{t \to 0} Y_1(t) = \lambda t \tag{6}$$

Since the process is time-homogeneous, we conclude that $\lambda \delta t$ is the probability that it goes through an increment in a small time interval δt and $1 - \lambda \delta t$ is the probability that it does not. By using the definition of $Y_i(t)$ and using the fact that $X(t)$ is time-homogeneous with independent increments we state that:

$$Y_i(t + \delta t) = (1 - \lambda \delta t) Y_i(t) + (\lambda \delta t) Y_{i-1}(t) \tag{7}$$

And directly from equation **(13)** we can say that:

$$\frac{Y_i(t+\delta t) - Y_i(t)}{\delta t} = -\lambda Y_i(t) + \lambda Y_{i-1}(t) \tag{8}$$

Now as $\delta t \to 0$, we get:

$$\frac{dY_i(t)}{dt} = -\lambda Y_i(t) + \lambda Y_{i-1}(t) \tag{9}$$

Now using the same arguments as **(2.84) to (2.89)** we state the required equality that $Y_i(t) = \frac{(\lambda t)^i}{i!} e^{-\lambda t}$.