Problem 7

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Problem 7 (2.5): Solve the equations for the birth process with:

$$\lambda_x = \lambda + yx, \;\; x = 0, 1, ... \tag{1}$$

and the initial conditions $P_1(0) = 1$**,** $P_x(0) = 0$**, for** $x \neq 1$**.**

Using the definition of λ_x in the birth process, we can directly, state that:

$$\frac{dP_x(t)}{dt} = -\lambda_x P_x(t) + \lambda_{x-1} P_{x-1}(t) \tag{2}$$

And using the equation (20) we can write that:

$$\frac{dP_x(t)}{dt} = -(\lambda + yx)P_x(t) + (\lambda + (x-1)y)P_{x-1}(t) \tag{3}$$

Substituting $x = 1$ we get:

$$\frac{dP_1(t)}{dt} = -(\lambda + y)P_1(t) \rightarrow P_1(t) = C_1 e^{-(\lambda+y)t} \tag{4}$$

And using the initial condition given in the problem, we can determine $P_1(t)$:

$$P_1(t) = e^{-(\lambda+y)t} \tag{5}$$

Now if we plug $x = 2$ in equation (22), we will get:

$$\frac{dP_2(t)}{dt} = -(\lambda + 2y)P_2(t) + (\lambda + y)P_1(t) \tag{6}$$

And by substituting the value of $P_1(t)$ we get:

$$\frac{dP_2(t)}{dt} = -(\lambda + 2y)P_2(t) + (\lambda + y)e^{-(\lambda+y)t} \tag{7}$$

Solving the ODE in equation (26) gives us the solution to $P_2(t)$:

$$P_2(t) = (1 + \frac{\lambda}{y})e^{-(\lambda+y)t}[1 - e^{-yt}] \tag{8}$$

Similarly, by using equation (22) we can plug $x = 3$ to get the ODE and then solve it for $P_3(t)$ to get:

$$P_3(t) = (1 + \frac{\lambda}{2y})(1 + \frac{\lambda}{y})e^{-(\lambda+y)t}[1 - e^{-yt}]^2 \tag{9}$$

In the very similar fashion we can solve for $P_4(t)$ to get:

$$P_4(t) = (1 + \frac{\lambda}{3y})(1 + \frac{\lambda}{2y})(1 + \frac{\lambda}{y})e^{-(\lambda+y)t}[1 - e^{-yt}]^3 \tag{10}$$

Now we make the following claim about $P_x(t)$:

$$P_x(t) = [\prod_{i=1}^{x-1}(1 + \frac{\lambda}{iy})]e^{-(\lambda+y)t}[1 - e^{-yt}]^{x-1} \tag{11}$$

Hence we will use induction to prove that the definition for $P_x(t)$ is the solution to equation (22). Hence assuming the solution true for $x = k$ we will prove it true for $x = k + 1$. Therefore, we need to show that:

$$\frac{dP_{k+1}(t)}{dt} = -(\lambda + (k + 1)y)P_{k+1}(t) + (\lambda + ky)P_k(t) \tag{12}$$

Putting $x = k + 1$ in equation (30) and differentiating, we get L.H.S.(31):

$$\frac{dP_{k+1}(t)}{dt} = -(\lambda + y)[\prod_{i=1}^{k}(1 + \frac{\lambda}{iy})]e^{-(\lambda+y)t}[1 - e^{-yt}]^k$$
$$+(\lambda + y)[\prod_{i=1}^{k-1}(1 + \frac{\lambda}{iy})]e^{-(\lambda+y)t}[1 - e^{-yt}]^{k-1}(e^{-yt})$$

Now we will solve the the Right-Hand-Side of the equation 31:

$$R.H.S.(31) = -(\lambda + (k + 1)y)[\prod_{i=1}^{k}(1 + \frac{\lambda}{iy})]e^{-(\lambda+y)t}[1 - e^{-yt}]^k$$
$$+(\lambda + ky)[\prod_{i=1}^{k-1}(1 + \frac{\lambda}{iy})]e^{-(\lambda+y)t}[1 - e^{-yt}]^{k-1}$$

We continue by adding and subtracting the same expression from RHS(31) and therefore, we get:

$$R.H.S.(31) = -(\lambda + (k + 1)y)[\prod_{i=1}^{k}(1 + \frac{\lambda}{iy})]e^{-(\lambda+y)t}[1 - e^{-yt}]^k$$
$$+(\text{ky})[\prod_{i=1}^{k}(1 + \frac{\lambda}{iy})]e^{-(\lambda+y)t}[1 - e^{-yt}]^k$$
$$+(\lambda + ky)[\prod_{i=1}^{k-1}(1 + \frac{\lambda}{iy})]e^{-(\lambda+y)t}[1 - e^{-yt}]^{k-1}$$
$$-(ky)[\prod_{i=1}^{k}(1 + \frac{\lambda}{iy})]e^{-(\lambda+y)t}[1 - e^{-yt}]^k$$

By solving, we get:

$$R.H.S.(31) = -(\lambda + y)[\prod_{i=1}^{k}(1 + \frac{\lambda}{iy})]e^{-(\lambda+y)t}[1 - e^{-yt}]^k$$
$$+(\lambda + y)[\prod_{i=1}^{k-1}(1 + \frac{\lambda}{iy})]e^{-(\lambda+y)t}[1 - e^{-yt}]^{k-1}(e^{-yt})$$

And we have proven that $LHS(31) = RHS(31)$, which shows that equation (30) is indeed the solution.