Problem 9

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Problem 9 (1.10): Consider a random walk process in which a moving particle can occupy any of the points $i = a+1, a+2, ..., b-1$ on the segment [a, b]. If the particle is at position i at time t, then it with probability p_i it will be at $i+1$ at time $t+1$ and with probability q_i it will be at $i-1$ at time $t+1$, where $p_i + q_i = 1$ for $i = a+1, a+2, ..., b-1$. Determine the probability, say $f(a;x)$ $x \in (a,b)$ that a particle starting at position x at $t = 0$ will land at the position a before landing at position b. Consider x and b to be fixed and a variable.

As stated in the problem, keeping b fixed, we introduce the function $f(a;x)$, defined for $a < x < b$, as the required probability that a particle starting at x lands at a before landing at b. In order to obtain the functional dependence upon a, we make the observation that the only way in which the particle can arrive at a before b is that it lands at $a+1$ before landing b and then from $a+1$ it goes to a before coming to b. This fact can be stated in the following functional equation **(9.1)**:

$$f(a;x) = f(a+1;x)f(a;a+1)$$

From (9.1) we obtain the equation **(9.2)**:

$$f(a;a+2) = f(a+1;a+2)f(a;a+1)$$

Combining with the relation **(9.3)**:

$$f(a;a+1) = p_{a+1} + q_{a+1}f(a;a+2)$$

We obtain the recurrence relation **(9.4)**:

$$u(a) = \frac{p_{a+1}}{1-q_{a+1}u(a+1)},$$

where $u(a) = f(a;a+1)$, valid for $a = b-3, b-4, ..., a+1$. It can be proven that $u(b-2) = p_{b-1}$ in the following way:

$$u(b-2) = \frac{p_{b-1}}{1-q_{b-1}u(b-1)}$$

And therefore,

$$u(b-1) = \frac{p_b}{1-q_b u(b)} = 0,$$

since $p_b = 0$. Thus the sequence $u(a)$ is determined.

Looking again at Equation (9.1), it follows that:

$$f(a; x) = \prod_{i=a}^{x-1} u(i)$$