Chap 4: Boundary Conditions

4.1 Dirichlet condition

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \\ u(0,t) = u(L,t) = 0 \end{cases}$$

$$U_n(x,t) = D_n \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right]$$

$$T(t) = C\cos(\sqrt{\lambda}ct) + D\sin(\sqrt{\lambda}ct)$$

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

Eigenvalues

$MX = \lambda X$

$L(X)$

$L(X)$

$(X'' - \lambda X = 0)$

$L : \mathbb{R}^n \to \mathbb{R}^n$

$L(X) = \lambda X$

If $\lambda = 0 \implies X'' = 0$

$X(x) = Ax + B \implies \boxed{A = B = 0}$ How?

$\lambda < 0 \qquad \lambda = -\beta$

$X'' + \beta X = 0$

$\Longrightarrow X(x) = Ae^{\sqrt{B}x} + Be^{-\sqrt{B}x}$

$X(0) = A + B = 0$

$X(L) = Ae^{\sqrt{B}L} + Be^{-\sqrt{B}L} = 0$

$\boxed{A = -B}$

$\Longrightarrow -Be^{\sqrt{B}L} = Be^{\sqrt{B}L}$ (↯)

$* \quad \lambda \in \mathbb{C} \qquad \lambda = i\beta$

$X'' - \lambda X = 0$

$X'' + i\beta X = 0 \Longrightarrow X = Ae^{i\sqrt{B}x} + Be^{-i\sqrt{B}x}$

$e^{i\sqrt{B}x} = \cos(\sqrt{B}x) + i(\sin(\sqrt{B}x))$

$e^{-i\sqrt{B}x} = \cos(\sqrt{B}x) - i\sin(\sqrt{B}x)$

$X = (A+B)\cos(\sqrt{B}x) + i(A-B)\sin(\sqrt{B}x)$

$X(0) = A + B = 0 \Longrightarrow A = -B$

$X(L) = i(A-B)\sin(\sqrt{B}L) = -2iB\sin(\sqrt{B}L)$

$\lambda = \sqrt{iB}$

Proof :

~~Suppo~~ ~~$\phi(x) = A_0$~~

Suppose ($\star$) statement is true.

$$\int_{-L}^{L} \phi(x)\cos\left(\frac{n\pi x}{L}\right)dx = \underbrace{A_0\int_{-L}^{L}\cos\left(\frac{k\pi x}{L}\right)}_{=0 \text{ (How?)}}$$

$$+\sum_{n=1}^{\infty}\left[A_n\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{k\pi x}{L}\right)dx\right.$$

$$\left.+B_n\int_{-L}^{L}\cos\left(\frac{k\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx\right] \quad \text{— } (\star\star)$$

$$\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{k\pi x}{L}\right)dx = 0 \quad \text{for all } k, n$$

$$\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{k\pi x}{L}\right)dx = 0 \quad \text{if } n \neq k \qquad \text{— (1)}$$

we know, $\cos(a)\cos(b) = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$

$$(1) \Rightarrow \underbrace{\frac{1}{2}\int \cos\left(\frac{(n+k)\pi x}{L}\right)dx}_{} + \frac{1}{2}\int_{-L}^{L}\cos\left(\frac{(n-k)\pi x}{L}\right)dx \longrightarrow 0$$

$$= \frac{1}{2}\,\frac{L}{\pi(n+k)}\left[\sin\left(\frac{(n+k)\pi x}{L}\right)\right]_{-L}^{L} = 0$$

also $(\sin a)(\sin b) = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$

$$\boxed{(\star\star) \Rightarrow A_k L \Rightarrow A_k = \frac{1}{L}\int_{-L}^{L}\phi(x)\frac{\cos(k\pi x)}{L}dx}$$

We know $\sqrt{i^2} = i \Rightarrow (i)^2 = -1$

$$i = e^{i\frac{\pi}{2}}$$

$$\Rightarrow (i)^{\frac{1}{n}} = e^{i\frac{\pi}{2n}}$$

$$(i)^{1/2} = e^{i\frac{\pi}{4}} =$$

Hence $\lambda = \sqrt{iB} = e^{i\frac{\pi}{4}}\sqrt{B}$

Chap 5. Fourier Series

$$f(x) = \sum_{n=0}^{+\infty} f_n x^n$$

$$f(x) = \sum_{n=0}^{+\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots \Rightarrow f(0) = a_0$$

$$f'(x) = \sum_{n=0}^{\infty} a_n n x^{n-1} = a_1 + 2a_2 + 3a_3 + 4a_4 + \dots$$

$$\Rightarrow f'(0) = a_1$$

Similarly, $f^{(n)}(0) = a_n (n!)$

Proposition

ϕ is a piecewise cont. function on $[-L, L]$

then $\phi(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)$ (A)

$$A_n = \frac{1}{L}\int_{-L}^{L} \phi(x)\cos\left(\frac{n\pi x}{L}\right) dx \qquad B_n = \frac{1}{L}\int_{-L}^{L} \phi(x)\sin\left(\frac{n\pi x}{L}\right) dx$$