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Probability Theory and Stochastic Calculus

Def. Let $0<\gamma<1$, A function $f:[0,T]\to\mathbb{R}$ is called uniformly Holder continuous with exponent $\gamma>0$ if there exist a constant K such that

$$|f(t)-f(s)|\leq K|t-s|^{\gamma}, \quad \forall s,t\in[0,T].$$

Theorem: Let $\{X\}$ be a stochastic process with continuous sample paths a.s., such that $E(|X(t)-X(s)|^{\beta})\leq C|t-s|^{1+\alpha}$

for constants $\beta,\alpha>0$, $C\geq 0$ and for all $0\leq t,s$.

Then for each $0<\gamma<\frac{\alpha}{\beta}$, $T>0$ and almost every ω, there exist a constant $K=K(\omega,\gamma,T)$ such that

$$|X(t,\omega)-X(s,\omega)|\leq K|t-s|^{\gamma}, \quad \forall s,t\leq T.$$

Application to Brownian Motion: For $m\in\mathbb{Z}^{+}$, $W(\cdot)$ n-dimentional

$$E(|W(t)-W(s)|^{2m}) = \frac{1}{(2\pi r)^{n/2}}\int_{R^n}|x|^{2m}e^{-\frac{|x|^2}{2r}}dx$$

$$= \frac{1}{(2\pi)^{n/2}}r^{m}\int_{R^n}|y|^{2m}e^{-\frac{|y|^2}{2}}dy \quad \left(y=\frac{x}{\sqrt{r}}\right)$$

$$= Cr^{m}$$

$$= C|t-s|^{m}$$

The hypothesis of the above theorem will hold for $\beta=2m$, $\alpha=m-1$. The process $W(\cdot)$ is thus Holder continuous a.s. for exponents

$$0<\gamma<\frac{\alpha}{\beta}=\frac{m-1}{2m}=\frac{1}{2}-\frac{1}{2m}$$

Hence for almost all ω and any $T>0$ the sample path $t \mapsto W(t,\omega)$ is uniformly Holder continuous on $[0,T]$ for each exponent $0<\gamma<\frac{1}{2}$.

Proof: For simplicity ~~take~~ $T=1$, and we pick any

$$0<\gamma<\frac{\alpha}{\beta} \quad \text{—— } (\star)$$

For $n=1,\dots$ ~~for~~ we define

$$A_n := \left\{ \left| X\left(\frac{i+1}{2^n}\right) - X\left(\frac{i}{2^n}\right) \right| > \frac{1}{2^{n\gamma}} \text{ for some integer } 0 \le i \le 2^n \right\}$$

We will show that $P(A_n) = 0$ as $n \to \infty$

$$\text{Then, } P(A_n) \le \sum_{i=0}^{2^n-1} P\left(\left| X\left(\frac{i+1}{2^n}\right) - X\left(\frac{i}{2^n}\right) \right| > \frac{1}{2^{n\gamma}} \right)$$

$$\le \sum_{i=0}^{2^n-1} E\left(\left| X\left(\frac{i+1}{2^n}\right) - X\left(\frac{i}{2^n}\right) \right|^{\beta} \right) \left(\frac{1}{2^{n\gamma}}\right)^{-\beta} \quad \text{(By Chebyshev)}$$

$$\le C \sum_{i=0}^{2^n-1} \left(\frac{1}{2^n}\right)^{1+\alpha} \left(\frac{1}{2^{n\gamma}}\right)^{-\beta}$$

$$= C\, 2^{n(-\alpha+\gamma\beta)}$$

However because of condition $(\star)$ $-\alpha+\gamma\beta<0$ we deduce that

$$\sum_{n=1}^{\infty} P(A_n) < \infty$$

By Borel-Cantelli Lemma $P(\limsup_{n\to\infty} A_n) = 0$. Hence $\exists\, m$ large enough

$$\left| X\left(\frac{i+1}{2^n}\right) - X\left(\frac{i}{2^n}\right) \right| \le \frac{1}{2^{n\gamma}}, \quad \text{for } 0 \le i \le 2^n - 1, \text{ given } n \ge m.$$

But then we have,

$$\left|X\left(\frac{i+1}{2^n}\right)-X\left(\frac{i}{2^n}\right)\right| \leq K\frac{1}{2^{n\gamma}}, \text{ for all } n\geq 0$$

if we select K large enough — (☆☆)

We will now ~~use this~~ prove that this fact ~~to prove the H~~ implies the stated Holder continuity. We fix $\omega\in\Omega$ and let $t_1, t_2 \in [0,1]$, $0<t_2-t_1<1$.

Select $n\geq 1$ so that $$\frac{1}{2^n} < t_2-t_1 < \frac{1}{2^{n-1}} \quad \text{—} (3☆)$$

We can write $$t_1 = \frac{i}{2^n}-\frac{1}{2^{p_1}}-\frac{1}{2^{p_2}}-\cdots-\frac{1}{2^{p_k}} \quad (n<p_1<\dots<p_k)$$

$$t_2 = \frac{j}{2^n}+\frac{1}{2^{q_1}}+\frac{1}{2^{q_2}}+\dots+\frac{1}{2^{q_\ell}} \quad (n<q_1<\dots<q_\ell).$$

for $$t_1\leq\frac{i}{2^n}\leq\frac{j}{2^n}\leq t_2 \Longrightarrow \frac{j-i}{2^n}\leq|t_2-t_1| < \frac{1}{2^{n-1}}$$

From (☆☆) $$\left|X\left(\frac{j}{2^n}\right)-X\left(\frac{i}{2^n}\right)\right| \leq K\left|\frac{j-i}{2^n}\right|^\gamma \leq K|t_2-t_1|^\gamma$$

But we have to show it for t_2 and t_1. Observe that

$$\left|X\left(\frac{i}{2^n}-\frac{1}{2^{p_1}}-\cdots-\frac{1}{2^{p_r}}\right)-X\left(\frac{i}{2^n}-\frac{1}{2^{p_1}}-\cdots-\frac{1}{2^{p_{r-1}}}\right)\right| \leq K\left|\frac{1}{2^{p_r}}\right|^\gamma$$

For $r=1,\dots,k$. Consequently,

$$\left|X(t_1,\omega)-X\left(\frac{i}{2^n},\omega\right)\right| \leq K\sum_{r=1}^{k}\left|\frac{1}{2^{p_r}}\right|^\gamma \leq \frac{K}{2^{n\gamma}}\sum_{r=1}^{\infty}\frac{1}{2^{r\gamma}} = \frac{C}{2^{n\gamma}} \leq C|t_2-t_1|^\gamma$$

(since $p_r>n$) (From 3☆).

In the same way we have deduced,

$$|X(t_2) - X(\tfrac{j}{2^n})| \leq C|t_2 - t_1|^\gamma$$

We add up the previous two to find out that,

$$|X(t_2) - X(t_1)| \leq C|t_2 - t_1|^\gamma$$

Since $t \mapsto X(t,w)$ is continuous for a.e w, the above estimate holds for all $t_1, t_2 \in [0,1]$.

Nowhere Differentiability

Theorem: ~~For~~ each $\frac{1}{2} < \gamma < 1$ and almost every ω, $t \mapsto W(t,\omega)$ is nowhere Holder Continuous with exponent γ.

Proof: We fix an integer N so large that $N(\gamma - \frac{1}{2}) > 1$. We only consider times $0 \leq t \leq 1$, given $0 \leq s \leq 1$, ~~then~~ if the function $t \mapsto W(t,\omega)$ is Holder Continuous with exponent γ, ~~such that~~ then

$$|W(t,\omega) - W(s,\omega)| \leq K|t-s|^\gamma \text{ for all } t \in [0,1] \text{ and some constant } K.$$

For $n >> 1$, set $i = \lfloor ns \rfloor + 1$ and note that for $j = i, i+1, \dots, i+N-1$

$$\begin{aligned}
|W(\tfrac{j}{n}) - W(\tfrac{j+1}{n})| &\leq |W(s) - W(\tfrac{j}{n})| \\
&\quad + |W(s) - W(\tfrac{j+1}{n})| \\
&\leq K\left(|s - \tfrac{j}{n}|^\gamma + |s - \tfrac{j+1}{n}|^\gamma\right) \\
&\leq \frac{M}{n^\gamma}
\end{aligned}$$

for some constant M.

Thus,

$$w \in A^i_{M,n} := \left\{ \left|W\left(\tfrac{j}{n}\right) - W\left(\tfrac{j+1}{n}\right)\right| \le \frac{M}{n^\gamma},\ \text{for } j = i, i+1 \cdots i+N-1 \right\}$$

for some $1 \le i \le n$, some $M \ge 1$ and all large n.

Therefore the set of $w \in \Omega$ such that $W(w,\cdot)$ is Holder continuous with exponent γ at some time $0 \le s \le 1$ is contained in,

$$\bigcup_{M=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \bigcup_{i=1}^{n} A^i_{M,n}$$

We will show that this event has probability 0.

For all k and M,

$$P\left(\bigcap_{n=k}^{\infty} \bigcup_{i=1}^{n} A^i_{M,n}\right) \le \liminf_{n\to\infty} P\left(\bigcup_{i=1}^{n} A^i_{M,n}\right)$$

$$\le \liminf_{n\to\infty} \sum_{i=1}^{n} P(A^i_{M,n})$$

$$\le \liminf_{n\to\infty} n\left(P\left(\left|W\left(\tfrac{1}{n}\right)\right| \le \frac{M}{n^\gamma}\right)\right)^N$$

Since the random variables $W\left(\frac{j+1}{n}\right) - W\left(\frac{j}{n}\right)$ are $N\left(0, \frac{1}{n}\right)$ and independents

$$P\left(\left|W\left(\tfrac{1}{n}\right)\right| \le \frac{M}{n^\gamma}\right) = \frac{\sqrt{n}}{\sqrt{2\pi}} \int_{-Mn^{-\gamma}}^{Mn^{-\gamma}} e^{-\frac{nx^2}{2}}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-Mn^{1/2-\gamma}}^{Mn^{1/2-\gamma}} e^{-y^2/2}\, dy$$

$(y = x\sqrt{n})$

$$\le C n^{1/2-\gamma}$$

Hence we deduce $P\left(\bigcap_{n=k}^{\infty} \bigcup_{i=1}^{n} A^i_{M,n}\right) \le \liminf_{n\to\infty} nC\left[n^{1/2-\gamma}\right]^N = 0$

Since $N(\gamma - \frac{1}{2}) > 1$. This holds for all k, M. Thus,

$$P\left(\bigcup_{M=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \bigcup_{i=1}^{n} A_{M,n}^{i} \right) = 0.$$