Convergence

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This topic is about covering the 3 notions of convergence.

There can be different forms of convergence such as:
$u_n : N \to R$
$u_n \to l$ as $(n \to \infty)$
For example, $f_n(x) = (-1)^n(x)$ does not converge (converges only for $x = 0$).

1 Pointwise Convergence

$\forall x \in [a, b],\ f_n(x) \to f(x)$ as $n \to \infty$

2 Uniform Convergence

$\text{Max}_{x\in[a,b]}|f_n(x) - f(x)| \to 0$ as $n \to \infty$, where $\text{Max}_{x\in[a,b]}|f_n(x) - f(x)|$ is independent of x.

3 L^2 Convergence

$L^2([a, b]) = \{f$ such that $\int_a^b |f(x)|^2 dx < +\infty\}$

$f_n(x)$ converges in L^2 to f if:

$$\int_a^b \left| f_n(x) - f(x) \right|^2 dx \to 0 \ as \ n \to \infty \tag{1}$$

Here again, $\int_a^b |f_n(x) - f(x)|^2 dx$ is independent of x.

4 Example

Take the example of the following function to study its convergence:

$$f_n(x) = (1 - x)x^{n-1},\ x \in (0, 1) \tag{2}$$

Observe that for Pointwise Convergence:

$$\sum_{n=1}^{N} f_n(x) = \sum_{n=1}^{N} x^{n-1} - x^n = 1 - x + x - ... + x^{N-1} - x^N \tag{3}$$

$$= 1 - x^N \to 1 \ as \ N \to \infty \tag{4}$$

For the Uniform Convergence we get:

$$Max_{x\in[0,1]} \left| \sum_{n=1}^{N} f_n(x) - f(x) \right| \to 0 \ as \ N \to \infty \tag{5}$$

$$\left| \sum_{n=1}^{N} f_n(x) - f(x) \right| = |1 - x^N - 1| = x^N \tag{6}$$

For the L^2 Convergence of $f_n(x) = (1-x)x^{n-1}$ we get:

$$\int_a^b \left| \sum_{n=1}^{N} f_n(x) - f(x) \right|^2 dx \tag{7}$$

$$\int_0^1 x^{2N} dx = \frac{1}{2N+1} \to 0 \ as \ N \to \infty \tag{8}$$

Therefore, $\sum f_n$ converges in L^2 to f.

5 Another Example

Now we will study the convergence of the following function:

$$f_n(x) = \frac{n}{1+n^2x^2} - \frac{n-1}{1+(n-1)^2x^2} \tag{9}$$

First we will study the Pointwise Convergence of Equation 9: Observe that:

$$\sum_{n=1}^{N} f_n(x) = \frac{N}{1+N^2x^2} \leq \frac{N}{N^2x^2} = \frac{1}{Nx^2} \to 0 \quad as \quad N \to +\infty \tag{10}$$

Hence our equation 10 shows the pointxise convergence of f. For the Uniform Convergence we take the sum of absolute differences and see that:

$$\sum_{n=1}^{N} \left| f_n(x) - f(x) \right| = \frac{N}{1+N^2x^2} = g_n(x) \tag{11}$$

By differentiating the expression we get in (11) we arrive at:

$$g_n'(x) = -\frac{2xN}{1+N^2x^2} \leq 0 \tag{12}$$

Now we see from (12) that g_n is a decreasing function of x and it will take its maximum value when $x = 0$. Therefore we state that:

$$Max_{x\in[0,1]}\left|\sum_{n=1}^{N} f_n(x) - f(x)\right| = N \to +\infty \ as \ N \to +\infty \tag{13}$$

Therefore we have a non-uniform convergence.

For the L^2 Convergence we take the following approach:

$$I = \int_0^L \left|\sum_{n=1}^{N} f_n(x) - f(x)\right|^2 dx = \int_0^L \left|\frac{N}{1+N^2x^2}\right|^2 dx = N^2 \int_0^L \frac{1}{1+N^2x^2} dx \tag{14}$$

We substitute $y = Nx \to dx = dy/N$ into (14) to get:

$$I = N \int_0^{NL} \frac{1}{(1+y^2)^2} dy \tag{15}$$

As $N \to \infty$

$$\frac{I}{N} = \int_0^{\infty} \frac{dx}{(1+y^2)^2} = c \ (c > 0) \tag{16}$$

As $c > 0$ and $N \to +\infty$ then $I \to +\infty$