2.4 : Diffusion on the whole line

(1) $\begin{cases} u_t - k u_{xx} = 0 & (x,t) \in R \times R^+ \\ u(x,0) = \phi(x) \end{cases}$ (Homogeneous Linear)

5. Invariance properties of solution of u of (1)

(a) $u(x-y,t)$ is still a solution for a fixed y.

(b) Differentiation invariance : ($u_t, u_{tt}, u_x, u_{xx}, u_{xxx}, u_{tx}$) are solutions

(c) Linear combination of solution is a solution

(d) Integration invariance : $\int_{-\infty}^{\infty} u(x-y,t)g(y)\,dy$ is a solution

[$g(y)$ is an integrable function].

∴ To differentiate k-times, we assume continuous derivatives up to the order k.

(e) Dilation Invariant $u(\sqrt{a}x, at)$ is still a solution.

We use pty (e) to construct a particular solution $Q(x,t)$. which does not have dilations.

$$\Longrightarrow \quad Q(\sqrt{a}x, at) = Q(x,t)$$

$g(p) = Q(x,t)$, where $p = \dfrac{x}{\sqrt{4kt}}$ or $p = \dfrac{x}{\sqrt{ct}}$

ODE

After finding Q we build a general solution.

We set the initial condition:

$$\begin{cases} Q(x,0)=1, & x>0 \\ Q(x,0)=0, & x<0 \end{cases}$$

$$Q_t = g'(p)\frac{\partial p}{\partial t} = g'(p)\left(-\frac{1}{2t}\cdot\frac{x}{\sqrt{4kt}}\right)$$

$$= -\frac{1}{2t}\, p g'(p)$$

$$Q_x = \cancel{p}\, g'(p)\frac{\partial p}{\partial x} = g'(p)\left[\frac{1}{\sqrt{4kt}}\right]$$

$$Q_{xx} = \frac{\partial}{\partial p} g'(p)\cdot\frac{\partial p}{\partial x}\cdot\frac{1}{\sqrt{4kt}} = \frac{1}{4kt} g''(p)$$

Now, $Q_t - kQ_{xx} = -\frac{1}{2t} p g'(p) - \frac{1}{4t} g''(p)$

$$= -\frac{1}{2t}\left[p g'(p) + \frac{1}{2} g''(p)\right] = 0$$

$$\Longrightarrow \frac{1}{2} g''(p) + p g'(p) = 0$$

By solving the ODE $\Longrightarrow$ $\ln|y| = -p^2 + c$

$$\boxed{y = ce^{-p^2}}$$

$$g'(p) = ce^{-p^2}$$

$$g(p) = c_1 \int e^{-p^2}\, dp$$

$$Q(x,t) = g(p) = g\left(\frac{x}{\sqrt{4kt}}\right)$$

$$= C_1 \int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2}\, dp + C_2$$

Determine C_1, C_2 from initial conditions... $Q(x,0)$

Claim:

$$\int_0^\infty e^{-p^2}\, dp = \frac{1}{2}\int_{-\infty}^\infty e^{-p^2}\, dp = \frac{\sqrt{\pi}}{2}$$

Proof

$$\left(\int_{-\infty}^\infty e^{-x^2}\, dx\right)^2 = \int_{-\infty}^\infty e^{-x^2}\, dx \cdot \int_{-\infty}^\infty e^{-y^2}\, dy = \iint_{\mathbb{R}^2} e^{-x^2-y^2}\, dx\,dy \quad (=I)$$

$$= \int\left(\int e^{-x^2} \cdot e^{-y^2}\, dy\right) dx = \int e^{-x^2}\left(\int e^{-y^2}\, dy\right) dx$$

$$= \int e^{-x^2}\, dx \int e^{-y^2}\, dy$$

Evaluating I:

$x = r\cos\theta \qquad y = r\sin\theta$

$$\text{Jacobian} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r$$

$$I = \int_0^{2\pi}\int_0^{\infty} r \cdot e^{-r^2}\, dr\, d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2} e^{-r^2}\Big|_0^{\infty}\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\, d\theta = \pi$$

So, $\displaystyle\int_{-\infty}^{\infty} e^{-p^2}\, dp = \sqrt{I} = \sqrt{\pi}$

But $\displaystyle\int_0^{\infty} e^{-p^2}\, dp = \frac{1}{2}\int_{-\infty}^{\infty} e^{-p^2}\, dp = \frac{\sqrt{\pi}}{2}$

$x>0$: $$\lim_{t\to 0} Q(x,t) = \underbrace{c_1\int_0^{\infty} e^{-p^2}\,dp + c_2}_{} = 1$$

$$c_1\frac{\sqrt{\pi}}{2} + c_2 = 1 \qquad \text{—— (1)}$$

$x<0$: $$\lim_{t\to 0} Q(x,t) = c_1\int_0^{-\infty} e^{-p^2}\,dp + c_2 = 0$$

$$= -c_1\frac{\sqrt{\pi}}{2} + c_2 \qquad \text{—— (2)}$$

Adding (1) and (2) we get $2c_2 = 1 \Rightarrow c_2 = \frac{1}{2}$

Hence $c_1 = \frac{2}{\sqrt{\pi}}$ $c_2 = \frac{1}{\sqrt{\pi}}$

$$\boxed{Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}}\int_0^{x/\sqrt{4kt}} e^{-p^2}\,dp \ , \quad t>0}$$

Now if we have the initial condition:

$$(1)\ \begin{cases} u_t - k u_{xx} = 0 \\ u(x,0) = \phi(x) \end{cases}$$

Let $S(x,t) = \frac{\partial Q}{\partial x}(x,t)$

Theorem: Given a function ϕ such that

$\lim_{|y|\to\infty} \phi(y) = 0$, then

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\,\phi(y)\,dy \ , \quad t>0$$

is a unique solution to problem (1).