PDE (3/March/2016)

$$S(x,t) = \frac{\partial Q}{\partial x}(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{\frac{-x^2}{4kt}}$$

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\,\phi(y)\,dy$$

is unique solution to

$$\begin{cases} u_t - k u_{xx} = 0 \\ u(x,0) = \phi(x) \end{cases}$$

at least should satisfy $\lim_{|y|\to\infty} \phi(y) = 0$ — (1)

We need to check the initial condition

is $u(x,0) = \phi(x)$

$$u(x,t) = \int_{-\infty}^{\infty} \frac{\partial}{\partial x} Q(x-y,t)\,\phi(y)\,dy$$

$$= -\int_{-\infty}^{\infty} \frac{\partial}{\partial y} Q(x-y,t)\,\phi(y)\,dy$$

(Integrate by parts)

$$= -Q(x-y,t)\,\phi(y)\Big]_{y=-\infty}^{y=\infty}$$

$$+ \int_{-\infty}^{\infty} Q(x-y,t)\frac{\partial}{\partial y}\phi(y)\,dy \quad \text{— (2)}$$

Due to (1) $\left[-Q(x-y,t)\,\phi(y)\right]_{-\infty}^{\infty} \to 0$. (If ϕ is nice enough)

$$\Rightarrow \quad u(x,0) = \int_{-\infty}^{\infty} Q(x-y,0)\,\phi'(y)\,dy \quad \text{[From (2)]}$$

We have $Q(x-y,0) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x < 0 \end{cases}$

$$u(x,0) = \int_{-\infty}^{x} \underbrace{Q(x-y,0)}_{\substack{1 \\ x>y}} \phi'(y)\,dy + \int_{x}^{\infty} \underbrace{Q(x-y,0)}_{\substack{0 \\ x<y}} \phi''(y)\,dy$$

$$= \int_{-\infty}^{x} Q(x-y,0)\,\phi'(y)\,dy = \int_{-\infty}^{x} \phi'(y)\,dy$$

$$= \left[\phi(y)\right]_{-\infty}^{x}$$

$$= \phi(x) - \phi(\infty)$$

$$= \phi(x) \quad [\text{Due to } (4)]$$

For uniecity we will prove that the solution is unique a finite interval $[-L, L]$ and since L is arbitrary, it applies to the whole line. $(-\infty, \infty)$.

$$S(x,t) = \frac{\partial}{\partial x} Q(x,t) \; \frac{1}{2\sqrt{\pi k t}} \, e^{-x^2/4kt} \quad (\sigma = \sqrt{2kt})$$

<u>Properties:</u>

For a small t : S has a tall spike
For a large t : S is flatter.

$$\int_{-\infty}^{\infty} S(x,t)\,dx = 1$$

After using the substitution $q = \dfrac{x}{\sqrt{4kt}}$

$$S(x,t) = S(q) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-q^2}\, dq = 1$$

$\max\limits_{|x|>\delta} S(x,t) \to 0$ as $t \to 0$, for any $\delta > 0$.

$$Q(x,t) = \frac{1}{2} + \int_0^x S(y,t)\,dy \qquad \left[\text{As } \frac{\partial}{\partial x} Q(x,t) = S(x,t)\right]$$

It can be expressed in terms of the statistical error function.

$$Q(x,t) = \frac{1}{2} + \frac{1}{2} \operatorname{Erf}\left(\frac{x}{\sqrt{4kt}}\right).$$

$$\text{where } \operatorname{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2}\,dp$$

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\,\phi(y)\,dy$$

2.5 : Comparison of Waves & Diffusions

Properties	Waves	Diffusions
(i) Speed of Propagation	Finite $\leq c$	Infinite
(ii) Singularities for $t>0$	Transported along characteristic	Lost immediately
(iii) Well-posed for $t>0$	Yes	Yes
(iv) Well-posed for $t<0$	Yes	No
(v) Max Principle	No	Yes
(vi) At $t \to \infty$	Conserved Energy	Energy dissipates to 0
(vii) Information	Transported	Lost gradually