First Order Linear Equations

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1 Derivatives are Local Operators

Let the operator $\mathcal{L}$ be defined as $\mathcal{L}(u) = u'(x_0, y_0)$. Hence we have:

$$\mathcal{L}(u) = \frac{du}{dx}(x_0, y_0) \tag{1}$$

(1) is a local operator since we only need information around x_0. The derivative with respect to x is formally written as:

$$\frac{du}{dx}(x_0, y_0) = \lim_{\epsilon \to 0} \frac{u(x_0 + \epsilon, y_0) - u(x_0, y_0)}{\epsilon} \tag{2}$$

For the sake of revision we can remind that the Chain Rule is states that:

$$\frac{d}{dt}\Big[f(g(t))\Big] = f'(g(t))g'(t) \tag{3}$$

The function $I(t)$ is defined as:

$$I(t) = \int_a^b f(x,t)dx \tag{4}$$

Differentiating (4) with respect to t with give:

$$\frac{dI}{dt} = \frac{d}{dt}\int_a^b f(x,t)dx = \int_a^b \frac{df}{dt}(x,t)dx \tag{5}$$

Theorem 1.1 : Assume that $f, \frac{df}{dt}$ are continuous on [a,b]x[c,d] then for all $t \in$ [c,d]:

$$I'(t) = \int_a^b \frac{df}{dt}(x,t)dx \tag{6}$$

Proof:

$$\begin{aligned} I'(t) = \frac{I(t,\epsilon) - I(t)}{\epsilon} &= \frac{1}{\epsilon}\int_a^b [f(x,t+\epsilon) - f(x,t)]dx \\ &= \int_a^b \frac{f(x,t+\epsilon) - f(x,t)}{\epsilon}dx \\ &= \int_a^b \frac{df}{dt}(x,t)dx \end{aligned} \tag{7}$$

We can take the limit because:
1) f is continuous
2) a, b do not depend on t
3) a, b are finite

Theorem 1.2 :

$$I'(t) = \int_{a(t)}^{b(t)} \frac{df}{dt}(x,t)dx + b'(t)f(b(t),t) - a'(t)f(a(t),t) \tag{8}$$

Proof:

$$I(t) = \int_a^b f(x,t)dx = F(b,t) - F(a,t) \tag{9}$$

$$\begin{aligned}
I'(t) &= \frac{dF}{dt}(b,t) - \frac{dF}{dt}(a,t) \\
&= \frac{d}{dt}\Big[\int_{a(t)}^{b(t)} f(x,t)dx\Big] \\
&= \frac{d}{dt}[F(b(t),t) - F(a(t),t)] \\
&= b'(t)\frac{dF}{dx}(b(t),t) + \frac{dF}{dt}(b,t) - a'(t)\frac{dF}{dx}(a(t),t) - \frac{dF}{dt}(a,t) \\
&= \int_{a(t)}^{b(t)} \frac{df}{dt}(x,t)dx + b'(t)f(b(t),t) - a'(t)f(a(t),t)
\end{aligned} \tag{10}$$

2 The Constant Coefficient Equation

In this section we will solve the following equation:

$$au_x + bu_y = 0 \tag{11}$$

where a and b are non-zero constants.

Observe that in equation (11), we can introduce a change of variables as $x' = ax - by$ and $y' = bx - ay$.
By applying the Chain Rule we get:

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} = au_{x'} + bu_{y'} \tag{12}$$

and

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} = bu_{x'} - au_{y'} \tag{13}$$

Therefore by combining (12) and (13) with (11) we get:

$$au_x + bu_y = (a^2 + b^2)u_{x'} \tag{14}$$

Since $a^2 + b^2 \neq 0$ the equation takes the form $u_{x'} = 0$ and we get the final solution as:

$$u = f(y') = f(bx - ay) \tag{15}$$

2.1 Example

In this example we will solve the PDE $4u_x - 3u_y = 0$ with the auxiliary condition $u(0, y) = y^3$.

By (15) we have $u(x, y) = f(-3x - 4y)$. Now setting $x = 0$ yields:

$$u(0, y) = y^3 = f(-4y) \tag{16}$$

Letting $z = -4y$ in (16) gives:

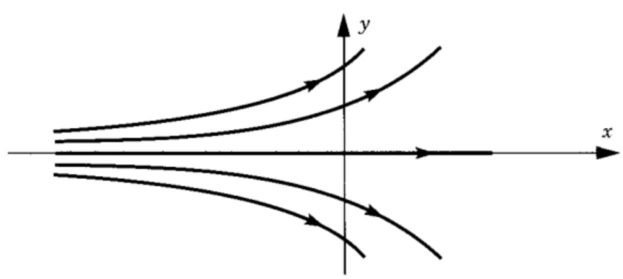
$$f(z) = -\frac{z^3}{64} \Rightarrow u(x, y) = \frac{(3x + 4y)^3}{64} \tag{17}$$

3 The Variable Coefficient Equation

In the previous section we covered the constant coefficient equation and in this one it will be about the variable coefficient equations. We will focus on solving the following equation:

$$u_x + yu_y = 0 \tag{18}$$

The curves in xy-plane with (1,y) as the tangent vector have slopes y (See Figure).



Therefore their equations are:

$$\frac{dy}{dx} = y \tag{19}$$

The ODE in (19) has the solution $y = Ce^x$. Therefore:

$$u(x, Ce^x) = u(0, Ce^0) = u(0, C) \tag{20}$$

Now putting $y = Ce^x$ in (20) we get the solution as:

$$u(x, y) = f(e^{-x}y) \tag{21}$$

3.1 Example

Solve the Partial Differential Equation:

$$\frac{\partial u}{\partial x} + 3x^2 y \frac{\partial u}{\partial y} = 0 \tag{22}$$

The characteristic curves of the PDE in (22) satisfy the ODE:

$$\frac{du}{dx} = 3x^2 y \frac{du}{dy} \Rightarrow \frac{dy}{dx} = 3x^2 y \tag{23}$$

By separating the variables and integrating we get:

$$\int \frac{1}{y} dy = \int 3x^2 dx \Rightarrow ln(y) = x^3 + C \Rightarrow y = e^{x^3 + C} \tag{24}$$

Again $u(x, y)$ is the constant on each curve. So to get the general solution of (22) we will have to solve (24) for $u(x, y) = f(C)$.

Hence we get the general solution as:

$$u(x, y) = f(ln(y) - x^3) \tag{25}$$