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7/April

$\boxed{\text{Fourier Series}}$

5.1- Coefficients

$f(x)$, $x \in [-L, L]$

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{xn\pi}{L}\right) + B_n \sin\left(\frac{xn\pi}{L}\right)$$

(or A_0)

$$A_n = \frac{1}{L}\int_{-L}^{L} \phi(x)\cos\left(\frac{n\pi x}{L}\right) dx$$

$$B_n = \frac{1}{L}\int_{-L}^{L} \phi(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

Example,

$\phi(x) = 1$, $x \in [-L, L]$

$$\phi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$A_0 = \frac{1}{L}\int_{-L}^{L} (1)\, dx = \frac{1}{L}(2L) = 2$$

$$\phi(x) = 1 + \sum \ldots$$

$$A_n = \frac{1}{L}\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{L} \times \frac{L}{n\pi}\left[\sin\left(\frac{n\pi x}{L}\right)\right]_{-L}^{L} \Rightarrow \boxed{A_n = \frac{1}{n\pi} \cdot 0}$$

$$B_n = \frac{1}{L}\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) dx = 0$$

$$\Rightarrow \phi(x) = 1$$

<u>Example,</u>

$$\phi(x) = x$$

$$A_0 = \frac{1}{L}\int_{-L}^{L} x\, dx = 0$$

$$A_n = \frac{1}{L}\int_{-L}^{L} x\cos\left(\frac{n\pi x}{L}\right) dx = 0$$

$$B_n = \frac{1}{L}\int_{-L}^{L} x\sin\left(\frac{n\pi x}{L}\right) dx = \frac{L}{n\pi} \times \frac{1}{L}\left[x\cos\left(\frac{n\pi x}{L}\right)\right]_{-L}^{L}$$

$$+ \underbrace{\frac{1}{n\pi}\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) dx}_{0}$$

$$\Rightarrow B_n = -\frac{2}{n\pi} L\cos(n\pi) = -\frac{2L}{n\pi}(-1)^n = \frac{2L}{n\pi}(-1)^{n+1}$$

Therefore,

$$\phi(x) = x = \sum_{n=1}^{\infty} \frac{2L}{n\pi} (-1)^{n+1} \sin\left(\frac{n\pi x}{L}\right)$$

Example

$$u_{tt} = c^2 u_{xx}$$

$$u(-L,t) = u(L,t) = 0$$

$$u(x,0) = x$$

$$u_t(x,0) = 0$$

$$\phi(x) = \sum_{n=1}^{+\infty} \frac{2L}{n\pi} (-1)^{n+1} \sin\left(\frac{n\pi x}{L}\right)$$

ϕ odd $\Rightarrow$ u odd

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right] D_n \sin\left(\frac{n\pi ct}{L}\right)$$

$$u_t(x,t) = \sum_{n=1}^{L} \frac{n\pi c}{L} \left[-A_n \sin\left(\frac{n\pi ct}{L}\right) + B_n \cos\left(\frac{n\pi ct}{L}\right)\right] D_n \sin\left(\frac{n\pi ct}{L}\right)$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n D_n \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} \frac{2L}{n\pi} (-1)^{n+1} \sin\left(\frac{n\pi x}{L}\right)$$

$$A_n D_n = \frac{2L}{n\pi} (-1)^{n+1}$$

$$U_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n D_n \sin\left(\frac{n\pi c}{L}\right) = 0$$

$$\Rightarrow B_n D_n = 0 \Rightarrow B_n = 0 \quad [\text{As } D_n \neq 0]$$

Therefore we get,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2L(-1)^{n+1}}{n\pi} \cos\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

Proposition

1) If f is odd

then $A_n = 0$ and $f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$

2) If f is even

then $B_n = 0$ and $f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$

Complex Form of the full Fourier Series

Theorem:

If $f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)$

then $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi x}{L}}$

$C_n = \frac{1}{2}\left[A_n - iB_n\right] \qquad C_{-n} = \overline{C_n} \qquad C_0 = \frac{A_0}{2}$

Proof:

$\cos\left(\frac{n\pi x}{L}\right) = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\left(\frac{n\pi x}{L}\right) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

$e^{i\theta} = \cos\theta + i\sin\theta$

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(\frac{A_n}{2} + \frac{B_n}{2i}\right)e^{i\theta} + e^{-i\theta}\left(\frac{A_n}{2} - \frac{B_n}{2i}\right)$$

$$\frac{A_n}{2} + \frac{B_n}{2i} \times \frac{i}{i} = \frac{A_n - iB_n}{2} = C_n$$

$$\frac{A_n}{2} - \frac{B_n}{2i} \times \frac{i}{i} = \frac{A_n + iB_n}{2} = \overline{C_n}$$

$$B_{-n} = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{-n\pi x}{L}\right)dx = -B_n$$

QED

$$\boxed{C_n = \frac{A_n - iB_n}{2}}$$

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} C_n e^{i\theta n} + C_{-n} e^{-i\theta n}$$

$$= \frac{A_0}{2} + \sum_{n=1}^{\infty} C_n e^{i\theta n} + \sum_{n=1}^{\infty} C_{-n} e^{-i\theta n}$$

$$= \frac{A_0}{2} + \sum_{k=1}^{\infty} C_k e^{i\theta k} + \sum_{k=-\infty}^{-1} C_k e^{i\theta k}$$

$$= \sum_{k=-\infty}^{\infty} C_k e^{i\theta k}$$

$$\boxed{C_n = \frac{1}{L} \int_{-L}^{L} f(x) e^{\frac{in\pi x}{L}} dx}$$

5.4 Completeness

1) 3 notions of convergence

$W_n : N \to R$ a sequence

$f_n(x) : N \times \mathbb{R} \to \mathbb{R}$ a sequence of functions

$W_n \longrightarrow L$ (as $n \to \infty$)

$\iff$ for $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ such that for all $n \geq N$
$|W_n - L| < \varepsilon \iff W_n \in (L - \varepsilon, L + \varepsilon)$

$f_n(x) \to f(x)$ Ex: $f_n(x) = (-1)^n x$, converges only for $x = 0$.