Markov Processes
Final Submission

Osama K Mahmood

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Problem 1 (1.1): Let $X_n, n = 0, 1, 2, ...$ be a sequence of independent random variables, each of which assumed non-negative integer values. Define a sequence of partial sums:

$$S_n = \sum_{i=1}^{n} X_i \tag{1}$$

Show that $S_n, n = 0, 1, 2, ...$ is a Markov Chain.

By Definition, we have:

$$S_n = \sum_{i=1}^{n} X_i = X_n + \sum_{i=1}^{n-1} X_i \tag{2}$$

$$X_n = S_n - S_{n-1} \tag{3}$$

We know that:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{4}$$

Therefore,

$$P(S_{n+1} = s_{n+1}|S_n = s_n, S_{n-1} = s_{n-1}, S_{n-2} = s_{n-2}, ..., S_1 = s_1) =$$

$$\frac{P(S_{n+1}=s_{n+1},S_n=s_n,S_{n-1}=s_{n-1},S_{n-2}=s_{n-2},...,S_1=s_1)}{P(S_n=s_n,S_{n-1}=s_{n-1},S_{n-2}=s_{n-2},...,S_1=s_1} =$$

$$\frac{P(S_n+X_{n+1}=s_{n+1},S_{n-1}+X_n=s_n,S_{n-2}+X_{n-1}=s_{n-1},...,S_1+X_2=s_2 S_1=X_1=s_1)}{P(S_{n-1}+X_n=s_n,S_{n-2}+X_{n-1}=s_{n-1},...,S_1+X_2=s_2 S_1=X_1=s_1)} =$$

$$\frac{P(X_{n+1}=s_{n+1}-s_n,X_n=s_n-s_{n-1},X_{n-1}=s_{n-1}-s_{n-2},...,X_2=s_2-s_1,X_1=s_1)}{P(X_n=s_n-s_{n-1},X_{n-1}=s_{n-1}-s_{n-2},...,X_2=s_2-s_1,X_1=s_1)} =$$

$$\frac{P(X_{n+1}=s_{n+1}-s_n)P(X_n=s_n-s_{n-1})P(X_{n-1}=s_{n-1}-s_{n-2})...P(X_2=s_2-s_1)P(X_1=s_1)}{P(X_n=s_n-s_{n-1})P(X_{n-1}=s_{n-1}-s_{n-2})...P(X_2=s_2-s_1)P(X_1=s_1)} =$$

$$P(X_{n+1} = s_{n+1} - s_n)$$

Replacing n by $n + 1$ in Equation (3) we get:

$$X_{n+1} = S_{n+1} - S_n$$

Hence X_{n+1} is the incremental value between S_n and S_{n+1}. Therefore:

$$P(X_{n+1} = s_{n+1} - s_n) = P(S_{n+1} = s_{n+1} | S_n = s_n)$$

Therefore, we get:

$$P(S_{n+1} = s_{n+1} | S_n = s_n) = P(S_{n+1} = s_{n+1} | S_n = s_n, S_{n-1} = s_{n-1}, S_{n-2} = s_{n-2}, ..., S_1 = s_1)$$

Hence, S_n is a Markov Chain.

Problem 2 (1.2): Prove the the following propositions: Two states i and j of a Markov Chain *communicate* if and only if $L_{ij} > 0$ and $L_{ji} > 0$.

Recall that, states i and j *communicate* if $\exists n, m$ such that $p_{ij}^{(n)} > 0$ and $p_{ji}^{(m)} > 0$ **(Def. on page 14)**.
$L_{ij} > 0$ implies that $\sum_{n=1}^{\infty} K_{ij}^{(n)} > 0$ by **(1.15) pg. 15**.

We state the following for a particular $n' \leq n$:

$$p_{ij}^{(n')} = \sum_{v=1}^{n'} K_{ij}^{(v)} p_{jj}^{(n'-v)} \tag{5}$$

$$\geq K_{ij}^{(n')} p_{jj}^{(0)} = K_{ij}^{(n')} \tag{6}$$

Since we know that $p_{ij}^{(n)} > 0$ and that $n' \leq n$ we can say that $K_{ij}^{(n')} > 0$. Hence we can be certain that, $L_{ij} > 0$. By symmetry we can prove that $L_{ji} > 0$.

Problem 3 (1.3): Classify the states of the Markov chains whose transition probabilities are given by:
(a) $p_{02} = 1; p_{11} = 1; p_{i,i-1} = p_{i,i+1} = \frac{1}{2}$

At first we will classify the states of the Markov Chain. Since (1) is an absorption state, it is a separate class.
Also because we cannot go from 2 to 0, (0) is a separate class. Hence following are the classes:

(0) (1) (2, 3, ...)

Since $p_{02} = 1$, (0) is a transient state.
Since $p_{11} = 1$, (1) is an absorption state.

For any state $i \in (2,3,...)$, Q=$(\frac{q}{p})^{i-1}$=1 **(from 1.118)**
According to the definition on **pg.15** we get $L_{ii} < 1$, and therefore i is a Transient State.

(b) $p_{00} = \frac{1}{2}; p_{01} = \frac{1}{2}; p_{i,i-1} = p_{i,i+1} = \frac{1}{2}$

Since we can move from any state i to any other state j, all the states are in the same class.
WLOG: $i < j$. $p_{ij}^{(j-i)} = (\frac{1}{2})^{(j-i)} \rightarrow p_{ij} > 0$. Therefore, there is only one class of states:

$$(0,\ 1,\ 2,\ ...)$$

Now for any given state $i \in (0,1,2,...)$,

$$Q = (\tfrac{q}{p})^i = 1 \quad \textbf{(from 1.118)}$$

Here Q represents the probability that the particle will ever (or eventually) reach the state 0. Therefore, 0 is a Recurrent state and since all the states are in the same class, they are all Recurrent.

(c) $p_{00} = \frac{1}{3}; p_{01} = \frac{2}{3}; p_{i,i-1} = \frac{1}{3}; p_{i,i+1} = \frac{2}{3}$

Since we can move from any state i to any other state j, all the states are in the same class.
If: $i < j$. $p_{ij}^{(j-i)} = (\frac{2}{3})^{(j-i)} \rightarrow p_{ij} > 0$.
If: $i > j$. $p_{ij}^{(i-j)} = (\frac{1}{3})^{(j-i)} \rightarrow p_{ij} > 0$. Therefore, there is only one class of states:

$$(0,\ 1,\ 2,\ ...)$$

Now for any given state $i \in (0,1,2,...)$,

$$Q = (\tfrac{q}{p})^i = (\tfrac{1}{2})^i < 1. \quad \textbf{(from 1.118)}$$

Here, Q has the same definition as in part (b). Therefore, 0 is a Transient state and since all the states are in the same class, they are all Transient.

Problem 4 (1.4): Calculate the higher moments and cumulants of X_n, where X_n is a simple branching process.

The Moment Generating Function of the random variable X is given by **(1.28)**:
$M(t) = E[e^{tX}]$ setting $s = e^t$, therefore:
$M(t) = E[1 + Xt + \frac{X^2}{2!}t^2 + \frac{X^3}{3!}t^3 + ...]$

Just as done in **Bharucha-Reid, pg. 20** we will compute the third cumulant in terms of the generating functions. As compiled in **(1.36)**, we get the following result **(4.1)**:

$i_3 = D^3(X_1) = F'''(1) - F''(1) - 3F'(1) - 3F'(1)F''(1) - 3[F'(1)]^2 + 2[F'(1)]^3$

And as in **(1.37)**, we get the following expression **(4.2)**:

$$D^3(X_n) = F_n'''(1) - F_n''(1) - 3F_n'(1) - 3F_n'(1)F_n''(1) - 3[F_n'(1)]^2 + 2[F_n'(1)]^3$$

Now we will proceed as we proceeded in equations **(1.38)** and **(1.39)**. By differentiating **(1.38)** we get **(4.3)**:

$$F_{n+1}'''(1) = F_n'''(1)[F'(1)]^3 + 3F_n''(1)F'(1)F''(1) + F_n'(1)F'''(1)$$

And by differentiating **(1.39)** we get **(4.4)**:

$$F_{n+1}'''(1) = F'''[F_n(1)][F_n'(1)]^3 + 3F''[F_n(1)]F_n'(1)F_n''(1) + F'[F_n(1)]F_n'''(1)$$

Using **(4.1)** we also find the following explicit expression for $F'''(1)$ **(4.5)**:

$$F'''(1) = i_3 + \sigma^2 + m^2 + 2m + 3m\sigma^2 + m^3$$

We equate **(4.3)** and **(4.4)** and we get the following expression **(4.6)**:

$$F_n'''(1) = (i_3 + \sigma^2 + m^2 + 2m + 3m\sigma^2 + m^3)\tfrac{m^{n-1}(m^{2n}-1)}{m^2-1}$$

The expression for $D^3(X_n)$ follows by putting the right expressions in **(4.2)**

$$D^3(X_n) =$$
$$F'''(1)\tfrac{m^{n-1}(m^{2n}-1)}{m^2-1} - (3m+1)\sigma^2 m^n \tfrac{m^n-1}{m^2-m} - m^n(m^n-1) - 3m^n - 3m^{2n} + 2m^{3n}$$

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Problem 5 (2.2): Let $X(t), t \geq 0$ be a time-homogeneous stochastic process with independent increments and let:

$$Y_i(t) = P([X(t) - X(0)] = i)$$

satisfy the conditions:

$$\lim_{t\to 0}\frac{Y_1(t)}{t} = \lambda > 0;\ \lim_{t\to 0}\frac{1 - Y_0(t) - Y_1(t)}{t} = 0 \qquad (7)$$

Show that:

$$Y_i(t) = \frac{(\lambda t)^i}{i!}e^{-\lambda t} \qquad (8)$$

for i=0, 1, ...

By adding up the two equations given in **(7)** we get:

$$\lim_{t\to 0}\frac{1 - Y_0(t)}{t} = \lambda \qquad (9)$$

Equating this result with the first equation in **(7)** we get:

$$Y_1(t) + Y_0(t) = 1 \tag{10}$$

From equation **(9)** we get:

$$\lim_{t\to 0}(1 - Y_0(t)) = \lambda t \to \lim_{t\to 0} Y_0(t) = 1 - \lambda t \tag{11}$$

By substituting **(10)** in **(11)** we get that:

$$\lim_{t\to 0} Y_1(t) = \lambda t \tag{12}$$

Since the process is time-homogeneous, we conclude that $\lambda\delta t$ is the probability that it goes through an increment in a small time interval δt and $1 - \lambda\delta t$ is the probability that it does not. By using the definition of $Y_i(t)$ and using the fact that $X(t)$ is time-homogeneous with independent increments we state that:

$$Y_i(t + \delta t) = (1 - \lambda\delta t)Y_i(t) + (\lambda\delta t)Y_{i-1}(t) \tag{13}$$

And directly from equation **(13)** we can say that:

$$\frac{Y_i(t + \delta t) - Y_i(t)}{\delta t} = -\lambda Y_i(t) + \lambda Y_{i-1}(t) \tag{14}$$

Now as $\delta t \to 0$, we get:

$$\frac{dY_i(t)}{dt} = -\lambda Y_i(t) + \lambda Y_{i-1}(t) \tag{15}$$

Now using the same arguments as **(2.84) to (2.89)** we state the required equality that $Y_i(t) = \frac{(\lambda t)^i}{i!}e^{-\lambda t}$.

Problem 6 (2.3): If F(s,t) is the generating function associated with the random variable X(t), show that F(1/s,t) is the generating function associated with -X(t).

We know that:

$$F(s,t) = \sum P_x(t)s^x \tag{16}$$

We introduce a new variable $Y(t)$ such that $Y(t) = -X(t)$. We write the generating function of $Y(t)$ derived from **(16)**:

$$M(s,t) = \sum P_y(t)s^y \tag{17}$$

By replacing y by $-x$ in **(17)** we get:

$$M(s,t) = \sum P_x(t)s^{-x} \tag{18}$$

We solve the problem by stating that:

$$M(s,t) = \sum P_x(t)(1/s)^x = F(1/s,t) \tag{19}$$

Thus we showed that the generating function associated with $Y(t) = -X(t)$ is the same as $F(1/s,t)$.

Problem 7 (2.5): Solve the equations for the birth process with:

$$\lambda_x = \lambda + yx, \;\; x = 0, 1, ... \tag{20}$$

and the initial conditions $P_1(0) = 1$, $P_x(0) = 0$, for $x \neq 1$.

Using the definition of λ_x in the birth process, we can directly, state that:

$$\frac{dP_x(t)}{dt} = -\lambda_x P_x(t) + \lambda_{x-1} P_{x-1}(t) \tag{21}$$

And using the equation (20) we can write that:

$$\frac{dP_x(t)}{dt} = -(\lambda + yx)P_x(t) + (\lambda + (x-1)y)P_{x-1}(t) \tag{22}$$

Substituting $x = 1$ we get:

$$\frac{dP_1(t)}{dt} = -(\lambda + y)P_1(t) \rightarrow P_1(t) = C_1 e^{-(\lambda+y)t} \tag{23}$$

And using the initial condition given in the problem, we can determine $P_1(t)$:

$$P_1(t) = e^{-(\lambda+y)t} \tag{24}$$

Now if we plug $x = 2$ in equation (22), we will get:

$$\frac{dP_2(t)}{dt} = -(\lambda + 2y)P_2(t) + (\lambda + y)P_1(t) \tag{25}$$

And by substituting the value of $P_1(t)$ we get:

$$\frac{dP_2(t)}{dt} = -(\lambda + 2y)P_2(t) + (\lambda + y)e^{-(\lambda+y)t} \tag{26}$$

Solving the ODE in equation (26) gives us the solution to $P_2(t)$:

$$P_2(t) = (1 + \frac{\lambda}{y})e^{-(\lambda+y)t}[1 - e^{-yt}] \tag{27}$$

Similarly, by using equation (22) we can plug $x = 3$ to get the ODE and then solve it for $P_3(t)$ to get:

$$P_3(t) = (1 + \frac{\lambda}{2y})(1 + \frac{\lambda}{y})e^{-(\lambda+y)t}[1 - e^{-yt}]^2 \tag{28}$$

In the very similar fashion we can solve for $P_4(t)$ to get:

$$P_4(t) = (1 + \frac{\lambda}{3y})(1 + \frac{\lambda}{2y})(1 + \frac{\lambda}{y})e^{-(\lambda+y)t}[1 - e^{-yt}]^3 \tag{29}$$

Now we make the following claim about $P_x(t)$:

$$P_x(t) = [\prod_{i=1}^{x-1}(1 + \frac{\lambda}{iy})]e^{-(\lambda+y)t}[1 - e^{-yt}]^{x-1} \tag{30}$$

Hence we will use induction to prove that the definition for $P_x(t)$ is the solution to equation (22). Hence assuming the solution true for $x = k$ we will prove it true for $x = k + 1$. Therefore, we need to show that:

$$\frac{dP_{k+1}(t)}{dt} = -(\lambda + (k+1)y)P_{k+1}(t) + (\lambda + ky)P_k(t) \tag{31}$$

Putting $x = k + 1$ in equation (30) and differentiating, we get L.H.S.(31):

$$\frac{dP_{k+1}(t)}{dt} = -(\lambda + y)[\prod_{i=1}^{k}(1 + \frac{\lambda}{iy})]e^{-(\lambda+y)t}[1 - e^{-yt}]^k$$
$$+(\lambda + y)[\prod_{i=1}^{k-1}(1 + \frac{\lambda}{iy})]e^{-(\lambda+y)t}[1 - e^{-yt}]^{k-1}(e^{-yt})$$

Now we will solve the the Right-Hand-Side of the equation 31:

$$R.H.S.(31) = -(\lambda + (k+1)y)[\prod_{i=1}^{k}(1 + \frac{\lambda}{iy})]e^{-(\lambda+y)t}[1 - e^{-yt}]^k$$
$$+(\lambda + ky)[\prod_{i=1}^{k-1}(1 + \frac{\lambda}{iy})]e^{-(\lambda+y)t}[1 - e^{-yt}]^{k-1}$$

We continue by adding and subtracting the same expression from RHS(31) and therefore, we get:

$$R.H.S.(31) = -(\lambda + (k+1)y)[\prod_{i=1}^{k}(1 + \frac{\lambda}{iy})]e^{-(\lambda+y)t}[1 - e^{-yt}]^k$$
$$+(\text{ky})[\prod_{i=1}^{k}(1 + \frac{\lambda}{iy})]e^{-(\lambda+y)t}[1 - e^{-yt}]^k$$
$$+(\lambda + ky)[\prod_{i=1}^{k-1}(1 + \frac{\lambda}{iy})]e^{-(\lambda+y)t}[1 - e^{-yt}]^{k-1}$$
$$-(ky)[\prod_{i=1}^{k}(1 + \frac{\lambda}{iy})]e^{-(\lambda+y)t}[1 - e^{-yt}]^k$$

By solving, we get:

$$R.H.S.(31) = -(\lambda + y)[\prod_{i=1}^{k}(1 + \frac{\lambda}{iy})]e^{-(\lambda+y)t}[1 - e^{-yt}]^k$$
$$+(\lambda + y)[\prod_{i=1}^{k-1}(1 + \frac{\lambda}{iy})]e^{-(\lambda+y)t}[1 - e^{-yt}]^{k-1}(e^{-yt})$$

And we have proven that $LHS(31) = RHS(31)$, which shows that equation (30) is indeed the solution.

Problem 8 (1.8): A Markov Chain with a Transition Matrix P is said to be periodic with period ω if $P^{a+\omega} = P^a$ and ω is the smallest positive integer with this property. Determine the limit matrix Π for periodic chains.

We are given that for a periodic matrix of period ω we have $P^{a+\omega} = P^a$.
We define the Limit Matrix as follows:

$$\Pi = lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} P^i$$

(from Theorem 1.9 on page 36).
At first we will solve the problem for a particular $n \in Z^+$ and then to take the limit matrix we will pass to the limit as $n \to \infty$. We get the following expression for n:

$$\Pi_n = \frac{1}{n} \sum_{i=1}^{n} P^i$$

$$\Pi_n = \frac{1}{n}([\frac{n}{\omega}](P^1 + P^2 + P^3 + ... + P^\omega) + P^1 + P^2 + P^3 + ...P^{n-[\frac{n}{\omega}]\omega})$$

Where $[X]$ is the largest integer greater than or equal to X.

$$\Pi_n = \frac{[\frac{n}{\omega}]}{n}(\sum_{i=1}^{\omega} P^i) + \frac{1}{n}(P^1 + P^2 + P^3 + ... + P^{n-[\frac{n}{\omega}]\omega})$$

We see that as $n \to \infty$

$$\Pi = \frac{1}{\omega}(\sum_{i=1}^{\omega} P^i)$$

Problem 9 (1.10): Consider a random walk process in which a moving particle can occupy any of the points $i = a+1, a+2, ..., b-1$ on the segment [a, b]. If the particle is at position i at time t, then it with probability p_i it will be at $i+1$ at time $t+1$ and with probability q_i it will be at $i-1$ at time $t+1$, where $p_i + q_i = 1$ for $i = a+1, a+2, ..., b-1$. Determine the probability, say $f(a;x)$ $x \in (a,b)$ that a particle starting at position x at $t = 0$ will land at the position a before landing at position b. Consider x and b to be fixed and a variable.

As stated in the problem, keeping b fixed, we introduce the function $f(a;x)$, defined for $a < x < b$, as the required probability that a particle starting at x lands at a before landing at b. In order to obtain the functional dependence upon a, we make the observation that the only way in which the particle can arrive at a before b is that it lands at $a+1$ before landing b and then from $a+1$ it goes to a before coming to b. This fact can be stated in the following functional equation **(9.1)**:

$$f(a;x) = f(a+1;x)f(a;a+1)$$

From (9.1) we obtain the equation **(9.2)**:

$$f(a;a+2) = f(a+1;a+2)f(a;a+1)$$

Combining with the relation **(9.3)**:

$$f(a;a+1) = p_{a+1} + q_{a+1}f(a;a+2)$$

We obtain the recurrence relation **(9.4)**:

$$u(a) = \frac{p_{a+1}}{1-q_{a+1}u(a+1)},$$

where $u(a) = f(a; a+1)$, valid for $a = b-3, b-4, ..., a+1$. It can be proven that $u(b-2) = p_{b-1}$ in the following way:

$$u(b-2) = \frac{p_{b-1}}{1-q_{b-1}u(b-1)}$$

And therefore,

$$u(b-1) = \frac{p_b}{1-q_b u(b)} = 0,$$

since $p_b = 0$. Thus the sequence $u(a)$ is determined.

Looking again at Equation (9.1), it follows that:

$$f(a; x) = \prod_{i=a}^{x-1} u(i)$$

Problem 10 (1.6): Prove the fundamental theorem concerning branching processes by utilizing the theory of absorption probabilities.

Let us denote the number of individuals in the n-th generation by $Z^{(n)}$. This is a random variable and can be defined by a Markov Chain. The Transition Probability p_{jk} is defined as the probability that $Z^{(n+1)} = k$ given that $Z^{(n)} = j$.We define the generating function as **(Bharucha-Reid, 1.28)**:

$$F(x) = \sum_{k=0}^{\infty} p_k x^k$$

Due to the assumption of statistical independence we have **(Bharucha-Reid, 1.29)**:

$$p_{jk} = \text{the coefficient of } x^k \text{ in } F^j(x)$$

From this we can say that if the original population size is r, then the generating function of the population size $Z^{(n)}$ in the n-th generation is given by $F_n^r(x)$, where **(Bharucha-Reid, 1.30)**:

$$F_1(x) = F(x),\ F_2(x) = F(F(x)),\ \dots\ ,\ F_{n+1}(x) = F(F_n(x))$$

Now we consider the absorption probability α_r that the population size will eventually be reduced to 0 if the initial population is r. This can only be possible if the absorption state has a non-zero probability. Then the absorption probability must be a solution of the infinite system of equations **(10.1)**:

$$\alpha_r = \sum_{v=1}^{\infty} p_{rv}\alpha_v + p_{r0}$$

Now we might see that we can get a solution of the form $\alpha_r = \lambda^r$. With the R.H.S. also expressed as $F^r(x)$, and therefore (10.1) implies that $\lambda^r = F^r(\lambda)$. We write this as **(10.2)**:

$$\lambda = F(\lambda)$$

If λ is a solution of (10.2), then $\alpha_r = \lambda^r$ is a solution of (10.1). We know this from **(Bharucha-Reid, 1.4.C, pg.25)** that $\lambda = 1$ is always a solution of (10.2), but if $F'(1) > 1$ there exists another root $\lambda < 1$. Then the previous solution will not be unique (look at Bharucha-Reid, figure 1.2 on pg. 26). In this case the smallest solution will determine the absorption probability.

Problem 11 (2.10): Consider a birth-and-death process with parameters λ_x and μ_x. Let T_n denote the time required for the random variable to increase from n to $n+1$ and let $T_n^* = E\{T_n\}$. T_n^* is the conditional expected time, conditioned upon non-absorption or extinction. Show that $T_n^* = \frac{1}{\lambda_n} + \frac{\mu_n}{\lambda_n} T_{n-1}^*$

The probability density function for the time t elapsing until the occurrence of the first event population size has reached n is defined by **(11.1)**:

$$f(t) = (\lambda_n + \mu_n)\exp[-(\lambda_n + \mu_n)t]$$

This event has a probability $\frac{\lambda_n}{\lambda_n+\mu_n}$ of being a birth and in this case the population will increase from n to $n+1$. It has a probability $\frac{\mu_n}{\lambda_n+\mu_n}$ of being a death and in this case the population will reduce from n to $n-1$. Now we must have another passage from $n-1$ to n and then one from passage from n to $n+1$. So we have the relation **(11.2)**:

$$T_n^* = \frac{\lambda_n}{\lambda_n+\mu_n}\frac{1}{\lambda_n+\mu_n} + \frac{\mu_n}{\lambda_n+\mu_n}\left(\frac{1}{\lambda_n+\mu_n} + T_{n-1}^* + T_n^*\right)$$

This solves to the desired equation that **(11.3)**:

$$T_n^* = \frac{1}{\lambda_n} + \frac{\mu_n}{\lambda_n} T_{n-1}^*$$

Problem 12 (2.13): The Coefficient of Variation of the random variable $X(t)$ is defined as:

$$\nu[X(t)] = \frac{D[X(t)]}{E[X(t)]}$$

the ration of standard deviation and mean of $X(t)$. Determine the Coefficient of Variation and its Asymptotic Behavior for (a) Poisson Process, (b) Simple Birth Process and (c) Birth-and-Death Process.

Following is the derivation of the Coefficient of Variation for

(a) Poisson Process
We know that (Bharucha-Reid, 2.94):

$$E[X(t)] = \lambda t$$

And that (Bharucha-Reid, 2.95):

$$D^2[X(t)] = \lambda t \Rightarrow D[X(t)] = \sqrt{\lambda t}$$

Therefore the Coefficient of Variation is computed as:

$$\nu[X(t)] = \frac{D[X(t)]}{E[X(t)]} = \frac{\sqrt{\lambda t}}{\lambda t} = \frac{1}{\sqrt{\lambda t}}$$

As $t \to \infty$:

$$\nu[X(t)] \to 0$$

(b) Simple Birth Process
We know that (Bharucha-Reid, 2.110):

$$E[X(t)] = e^{\lambda t}$$

And that (Bharucha-Reid, 2.111):

$$D^2[X(t)] = e^{\lambda t}(e^{\lambda t} - 1) \Rightarrow D[X(t)] = e^{(\lambda t)/2}\sqrt{e^{\lambda t} - 1}$$

Therefore the Coefficient of Variation is computed as:

$$\nu[X(t)] = \frac{D[X(t)]}{E[X(t)]} = \frac{e^{(\lambda t)/2}\sqrt{e^{\lambda t}-1}}{e^{\lambda t}} = \sqrt{1 - e^{-\lambda t}}$$

As $t \to \infty$:

$$\nu[X(t)] \to 1$$

(c) Birth-and-Death Process
We know that (Bharucha-Reid, 2.154):

$$E[X(t)] = e^{(\lambda-\mu)t}$$

And that (Bharucha-Reid, 2.155):

$$D^2[X(t)] = \frac{\lambda+\mu}{\lambda-\mu}e^{(\lambda-\mu)t}(e^{(\lambda-\mu)t}-1) \Rightarrow D[X(t)] = \sqrt{\frac{\lambda+\mu}{\lambda-\mu}}e^{((\lambda-\mu)t)/2}\sqrt{e^{(\lambda-\mu)t} - 1}$$

Therefore the Coefficient of Variation is computed as:

$$\nu[X(t)] = \frac{D[X(t)]}{E[X(t)]} = \frac{\sqrt{\frac{\lambda+\mu}{\lambda-\mu}}e^{((\lambda-\mu)t)/2}\sqrt{e^{(\lambda-\mu)t}-1}}{e^{(\lambda-\mu)t}} = \sqrt{\frac{\lambda+\mu}{\lambda-\mu}(1 - e^{-(\lambda-\mu)t})}$$

As $t \to \infty$:

$$\nu[X(t)] \to \sqrt{\frac{\lambda+\mu}{\lambda-\mu}}$$