Chapter 3: Reflection & Sources

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$$\begin{cases} u_t = k u_{xx} \\ u(0,t) = 0 \end{cases} \quad u(x,0) = \phi(x) \qquad x \in [0, \infty) \text{ for all } t \geq 0$$

$$V_t - k V_{xx} = 0$$

$$V(x,0) = \psi$$

3.3

$$\begin{cases} u_t - k u_{xx} = f(x,t) \\ u(x,0) = \phi(x) \end{cases} \qquad x \in \mathbb{R}$$

Theorem

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy + \int_0^t \int_{-\infty}^{+\infty} S(x-y,t-s) f(y,s)\,dy\,ds$$

Proof

$$u_t - k u_{xx} = 0$$

$$S_t - k S_{xx} = 0 \qquad S(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4t}}$$

$\therefore u$ is a solution then $u_x, u_{xx} \ldots$ are also solutions.

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left[\int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s) f(y,s)\,dy\,dy \right]$$

$$= \int_{-\infty}^{\infty} \frac{\partial S}{\partial t}(x-y,t)\,\phi(y)\,dy$$

$$+ \int_0^t \int_{-\infty}^{+\infty} \frac{\partial S}{\partial t}(x-y,t-s)\,f(y,s)\,dy\,ds$$

$$+ \underbrace{\int_{-\infty}^{\infty} S(x-y,0)\,f(y,\cancel{t})\,dy}_{f(x,t)} \quad \text{— (Claim)}$$

$$\boxed{\frac{\partial}{\partial t}\left[\int_0^t g(x,t)\,dx\right] = g(x,t) + \int_0^t \frac{\partial g}{\partial t}(x,t)\,dx}$$

$$\boxed{\frac{\partial}{\partial x}\left[\int_0^x g(s)\,ds\right] = g(x)}$$

Proof

$$S(x-y,t) = \frac{1}{\sqrt{4\pi t}}\,e^{-\frac{|x-y|^2}{4t}} \xrightarrow[t \to 0]{x \neq y} 0$$

if $x=y$

if $x = y \Longrightarrow S(x-y,t) \longrightarrow 1$

$$\int_{-\infty}^{\infty} S(x-y,\varepsilon)\,f(y,t)\,dy$$

$$= \int_{x \neq y} S(x-y,\varepsilon) f + \int_{x=y} S(x-y,\varepsilon) f \longrightarrow f(x,t)$$

$\longrightarrow 0$ as $\varepsilon \longrightarrow 0$; $\longrightarrow 1$ as $\varepsilon \to 0$

Using $\frac{\partial S}{\partial t} = k \frac{\partial^2 S}{\partial x^2}$

$$\Rightarrow \quad \frac{\partial u}{\partial t} = k \int_{-\infty}^{\infty} \frac{\partial^2 S}{\partial x^2}(x-y,t)\,\phi(y)\,dy$$

$$+ k \int_0^t \int_{-\infty}^{\infty} \frac{\partial^2 S}{\partial x^2}(x-y,t-s)\,f(y,s)\,dy\,ds + f(x,t)$$

$$= k \frac{\partial^2 S}{\partial x^2} \underbrace{\left[\int_{-\infty}^{+\infty} S(x-y,t)\,\phi(y)\,dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)\,f(y,s)\,dy\,ds \right]}_{u(x,t)} + f(x,t)$$

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 S}{\partial x^2} u(x,t) + f(x,t)$$

$$u_t - k u_{xx} = f(x,t)$$

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\,\phi(y)\,dy + \int_0^t \int_{-\infty}^{+\infty} S(x-y,t-s)\,f(y,s)\,dy\,ds$$

as $t \to 0$

$u(x,0) = \phi$

$$u' + au = f$$

$$u' + au = 0 \implies u = e^{-au}$$

$$\cancel{e^{ax} u' + a e^{ax} u = f e^{ax}}$$

$$e^{at} u' + a e^{at} u = f e^{at}$$

$$u e^{at} - u(0) = \int_0^t f(s) e^{at} ds$$

$$u(t) = e^{-at} u(0) + \int_0^t f(s) e^{-a(t-s)} ds$$

3.4 : Wave with a Source

$$(1) \begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & x \in \mathbb{R} \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

Theorem:

The unique solution of (1) is

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds + \frac{1}{2c}\iint_{\Delta} f$$

$$Lu = u_{tt} - c^2 u_{xx} = (\partial_t + c\partial_x)(\partial_t u - c\partial_x u)$$

$$y = x + ct \qquad \xi = x - ct$$

$$Lu = -4c^2 u_{y\xi} = f(x,t)$$

$\boxed{\text{(f) Discuss Understand this}}$