International Mathematics Competition 2015

Problem-Solving Session

11-April-2015, E-047

Basic Algebraic Inequalities

Basic Number Theory

Organized By:

NYUAD Mathematics Club

A. Problems on Algebraic Inequalities

1. Triangular Inequality

   Prove that for all positive real numbers a, b, c we have

$$\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \geq \sqrt{a^2 + ac + c^2}.$$

2. AM-GM Inequality

   (India, 2002) *If a, b, c are positive real numbers, prove that*

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{c+a}{c+b} + \frac{a+b}{a+c} + \frac{b+c}{b+a}.$$

3. Cauchy-Schwarz Inequality

   Let a_1, a_2, $\ldots$, a_n be positive numbers such that $a_1 + a_2 + \cdots + a_n = 1$. Prove that

$$\frac{a_1}{\sqrt{1-a_1}} + \cdots + \frac{a_n}{\sqrt{1-a_n}} \geq \frac{1}{\sqrt{n-1}}(\sqrt{a_1} + \cdots + \sqrt{a_n}).$$

4. Jensen Inequality

   (China, 1989) *Prove that for any n real positive numbers x_1, $\ldots$, x_n such that $\sum_{i=1}^{n} x_i = 1$, we have*

$$\sum_{i=1}^{n} \frac{x_i}{\sqrt{1-x_i}} \geq \frac{\sum_{i=1}^{n} \sqrt{x_i}}{\sqrt{n-1}}.$$

5. Power-Mean Inequality

(Russia, 1999) The positive real numbers x and y satisfy $x^2 + y^3 \geq x^3 + y^4$. Prove that

$$x^3 + y^3 \leq 2.$$

6. Chebyshev Inequality

Let $a, b, c, d \in \mathbb{R}^+$ with $ab + bc + cd + da = 1$, prove that

$$\frac{a^3}{b+c+d} + \frac{b^3}{a+c+d} + \frac{c^3}{a+b+d} + \frac{d^3}{a+b+c} \geq \frac{1}{3}.$$

B. Inequalities to follow-by
(With Solutions)

Problem 1

Let a, b, c, d be real numbers with $a+d=b+c$, prove that

$$(a-b)(c-d)+(a-c)(b-d)+(d-a)(b-c)\geq 0.$$

Solution

$$(a-b)(c-d)+(a-c)(b-d)+(d-a)(b-c)=2(a-b)(c-d)=2(a-b)^2\geq 0.$$

Problem 2

If a, b, $c>0$ satisfy that $abc=1$, prove that

$$\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+ac}{1+c}\geq 3.$$

Solution

$$\tfrac{1+ab}{1+a}=\tfrac{abc+ab}{1+a}=ab\left(\tfrac{1+c}{1+a}\right),$$

$$\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+ca}{1+c}=ab\left(\frac{1+c}{1+a}\right)+bc\left(\frac{1+a}{1+b}\right)+ca\left(\frac{1+b}{1+c}\right)$$
$$\geq 3\sqrt[3]{(abc)^2}=3.$$

Problem 3

Let $x_1, x_2, \ldots, x_n>0$ such that $\frac{1}{1+x_1}+\cdots+\frac{1}{1+x_n}=1$. Prove that

$$x_1x_2\cdots x_n\geq(n-1)^n.$$

Solution

Setting $y_i = \frac{1}{1+x_i}$, then $x_i = \frac{1}{y_i} - 1 = \frac{1-y_i}{y_i}$. Observe that $y_1 + \cdots + y_n = 1$ implies that $1 - y_i = \sum_{j \neq i} y_i$, then $\sum_{j \neq i} y_i \geq (n-1)\left(\prod_{j \neq i} y_j\right)^{\frac{1}{n-1}}$ and

$$\prod_i x_i = \prod_i \left(\frac{1-y_i}{y_i}\right) = \frac{\prod_i \left(\sum_{j \neq i} y_j\right)}{\prod_i y_i} \geq \frac{(n-1)^n \prod_i \left(\prod_{j \neq i} y_j\right)^{\frac{1}{n-1}}}{\prod_i y_i} = (n-1)^n.$$

Problem 4

Let a, b, c be positive real numbers, prove that

$$(a+b)(a+c) \geq 2\sqrt{abc(a+b+c)}.$$

Solution

Dividing both sides of the given inequality by a^2 and setting $x = \frac{b}{a}$, $y = \frac{c}{a}$, the inequality becomes

$$(1+x)(1+y) \geq 2\sqrt{xy(1+x+y)}.$$

Now, dividing both sides by xy and making the substitution $r = 1 + \frac{1}{x}$, $s = 1 + \frac{1}{y}$, the inequality we need to prove becomes

$$rs \geq 2\sqrt{rs - 1}.$$

This last inequality is equivalent to $(rs-2)^2 \geq 0$, which become evident after squaring both sides and doing some algebra.

C. Number Theory Problems

Problem 1 (Divisibility and Modulo)

Let n be a positive integer not divisible by 2 or 3. Prove that for all integers k, the number $(k+1)^n - k^n - 1$ is divisible by $k^2 + k + 1$.

Problem 2 (Divisibility and Squares)

Determine all positive integers n such that $3^n + 1$ is divisible by n^2.

Problem 3 (Fermat's Little Theorem)

Find all triples (x, y, z) of positive integers such that $x \leq y \leq z$ and

$$x^3(y^3 + z^3) = 2012(xyz + 2).$$

Problem 4 (IMC-2014/D1/1)

Determine all pairs (a, b) of real numbers for which there exists a unique symmetric 2×2 matrix M with real entries satisfying $\text{trace}(M) = a$ and $\det(M) = b$.

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