International Mathematics Competition 2015

Problem-Solving Session

18-April-2015, E-047

IMC Problems

Organized By:

NYUAD Mathematics Club

Practice IMC Problems

Problem 1. Determine all pairs (a, b) of real numbers for which there exists a unique symmetric 2×2 matrix M with real entries satisfying $\operatorname{trace}(M) = a$ and $\det(M) = b$.

(Proposed by Stephan Wagner, Stellenbosch University)

Problem 2. For a positive integer x, denote its n^{th} decimal digit by $d_n(x)$, i.e. $d_n(x) \in \{0, 1, \ldots, 9\}$ and $x = \sum_{n=1}^{\infty} d_n(x)10^{n-1}$. Suppose that for some sequence $(a_n)_{n=1}^{\infty}$, there are only finitely many zeros in the sequence $\left(d_n(a_n)\right)_{n=1}^{\infty}$. Prove that there are infinitely many positive integers that do not occur in the sequence $(a_n)_{n=1}^{\infty}$.

(Proposed by Alexander Bolbot, State University, Novosibirsk)

Problem 3. Let z be a complex number with $|z+1| > 2$. Prove that $|z^3 + 1| > 1$.

(Proposed by Walther Janous and Gerhard Kirchner, Innsbruck)

Problem 4. Prove that for any pair of positive integers k and n there exist k positive integers $m_1, m_2, \ldots, m_k$ such that

$$1 + \frac{2^k - 1}{n} = \left(1 + \frac{1}{m_1}\right)\left(1 + \frac{1}{m_2}\right)\cdots\left(1 + \frac{1}{m_k}\right).$$

(Japan)

Problem 5. Consider the real numbers $x_0 > x_1 > x_2 > \cdots > x_n$. Prove that

$$x_0 + \frac{1}{x_0 - x_1} + \frac{1}{x_1 - x_2} + \cdots + \frac{1}{x_{n-1} - x_n} \geq x_n + 2n.$$

When does equality hold?