Introduction to Partial Differential Equations

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1 The Wave Equation

The Wave Equation is one of the most fundamental PDEs. It is written as follows:

$$\frac{d^2u}{dt^2} = c^2\frac{d^2u}{dx^2} \tag{1}$$

The equation $u'' + u = 0$ might be written as $\mathcal{L}(u) = 0$, where $\mathcal{L}$ is a linear operator. If A is a matrix then:

$$U' + AU = 0 \Rightarrow U(t) = e^{-tA}U_0 \tag{2}$$

The solution can be expressed in power series by the following fact:

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \tag{3}$$

And if in (2), we do not have a matrix coefficient then it is well-known that:

$$u' + u = 0 \Rightarrow u(x) = ce^{-x} \tag{4}$$

1.1 Example

The following is an example of an ODE:

$$-u'' + u' + \frac{3}{4}u = 0 \Rightarrow u(x) = c_1e^{\frac{3}{2}x} + c_2e^{-\frac{x}{2}} \tag{5}$$

Now for the solution given in (5), the Dimension of the solution is 2
$\Rightarrow$ dim(E)=2.
And if we have the following case:

$$a_nu^{(n)}(x) + a_{n-1}u^{(n-1)}(x) + ... + a_1u(x) = 0 \tag{6}$$

Then we would say that in (6), dim(E)=n.

1.2 Complex Numbers

Sometimes we also get an auxiliary equations that are solved in complex forms. Consider the following differential equation:

$$\frac{d^2u}{dx^2} + u = 0 \tag{7}$$

Now thw auxiliary equation of (7) will be $\alpha^2 + 1 = 0 \Rightarrow \alpha = \pm i$. We get the following solution:

$$u(x) = c_1 e^{ix} + c_2 e^{-ix} \tag{8}$$

Using the well-known Euler's Formula:

$$e^{ix} = \cos(x) + i\sin(x) \tag{9}$$

We arrive at:

$$u(x) = (c_1 + c_2)\cos(x) + i(c_1 - c_2)\sin(x) \tag{10}$$

Now in (10), we have $\dim_{\mathbb{C}}(E)$=2 and $\dim_{\mathbb{R}(E)}$=4.

2 Linear Differential Operators

If $\mathcal{L}$ is a Linear Differential Operator of Order j then:

$$\mathcal{L}_j : C^k(\mathbb{R}) \to C^{k-j}(\mathbb{R}) \tag{11}$$

The Linear Differential Operators have two main properties:
(1) $\mathcal{L}_j(u + v) = \mathcal{L}_j(u) + \mathcal{L}_j(v)$
(2) $\mathcal{L}_j(cu) = c\mathcal{L}_j(u)$

2.1 Examples

We can define an $\mathcal{L}$ as follows:

$$\mathcal{L}_2(u) = (u)_{xx} + (u)_{yy} + (u)_{xy} \tag{12}$$

Now we can test for property (1) for the operator in (12) that:

$$\mathcal{L}_2(u + v) = (u + v)_{xx} + (u + v)_{yy} + (u + v)_{xy} = \mathcal{L}_2(u) + \mathcal{L}_2(v) \tag{13}$$

But if we consider the following operator:

$$\mathcal{L}_2(u) = u_{xx} + u.u_y \tag{14}$$

Testing for property (1) in (14), we get:

$$\mathcal{L}_2(u + v) = \mathcal{L}_2(u) + \mathcal{L}_2(v) + u.v_y + v.u_y \neq \mathcal{L}_2(u) + \mathcal{L}_2(v) \tag{15}$$

Hence the operator in (14) is not linear.