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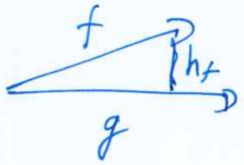
L^2 Theory

$$(f,g)_{L^2} = \int fg$$

$$\|f\|_{L^2} = \sqrt{(f,f)_{L^2}}$$

$$\text{dist}(f,g) = \|f-g\|_{L^2}$$



$$\text{proj}_g f = (f,g)_{L^2} \frac{g}{\|g\|_{L^2}}$$

$$\|h_f\| = \left\| f - (f,g)\frac{g}{\|g\|_{L^2}} \right\|_{L^2}$$

Proposition: $e^{\frac{inx\pi}{L}} = X_n(x)$ are a orthonormal basis of $L^2(-L,L)$

$n \neq j$ $\quad (X_n, X_j) = \int_{-L}^{L} X_n(x) X_j(x)\, dx = \int_{-L}^{L} e^{\frac{i(n+j)x\pi}{L}}\, dx = \frac{L}{i(n+j)\pi}\left[e^{\frac{i(n+j)x\pi}{L}}\right]_{-L}^{L}$

$$= \frac{L}{i(n+j)\pi}\left[e^{i(n+j)\pi} - e^{-i(n+j)\pi}\right] = \frac{2L}{(n+j)\pi}\sin\left[(n+j)\pi\right] = 0$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

If $n=j$

$$(X_n, X_n) = \int_{-L}^{L} e^{\frac{i2nx\pi}{L}} dx = \frac{L}{i2n\pi}\left[e^{\frac{i2nx\pi}{L}}\right]_{-L}^{L} = 0$$

$$(X_n, X_j)_{L^2} = \int_{-L}^{L} e^{\frac{inx\pi}{L}} e^{\frac{ijx\pi}{L}} dx$$

$$= \int_{-L}^{L} \left(\cos\left(\frac{nx\pi}{L}\right) + i\sin\left(\frac{nx\pi}{L}\right)\right)\left(\cos\left(\frac{jx\pi}{L}\right) + i\sin\left(\frac{jx\pi}{L}\right)\right) dx$$

$$= i\int_{-L}^{L} \cos\left(\frac{nx\pi}{L}\right)\sin\left(\frac{jx\pi}{L}\right) dx + \int_{-L}^{L} \cos\left(\frac{nx\pi}{L}\right)\cos\left(\frac{jx\pi}{L}\right) - \int_{-L}^{L} \sin\left(\frac{nx\pi}{L}\right)\sin\left(\frac{jx\pi}{L}\right) dx$$

$$+ i\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{j\pi x}{L}\right) dx = 0$$

$$\|X_n\|_{L^2}^2 = \int_{-L}^{L} \left(\cos\left(\frac{n\pi x}{L}\right) + i\sin\left(\frac{n\pi x}{L}\right)\right)^2 dx$$

$$= \int_{-L}^{L} \cos^2\left(\frac{n\pi x}{L}\right) dx - \int_{-L}^{L} \sin^2\left(\frac{n\pi x}{L}\right) dx \neq 0$$

Completeness

V a vector space

$\dim V = n$

$(e_1, e_2, \dots, e_n)$ ~~orthonormal~~ orthogonal basis

$$f \in V \quad f = \sum_{j=1}^{n} f_j e_j \qquad f_j = \frac{(f, e_j)}{\| e_j \|^2}$$

Proof:

$$(f, e_i) = \sum_{j=1}^{n} f_j (e_j, e_i)$$

if $i \neq j$ $(e_j, e_i) = 0$

if $i = j$ $(e_i, e_i) = \| e_i \|^2$

$$(f, e_i) = \sum_{j=1}^{n} f_j (e_j, e_i) = f_i (e_i, e_i) = f_i \| e_i \|^2$$

$$\Rightarrow \quad f_i = \frac{(f, e_i)}{\| e_i \|^2}$$

If V is a vector space (∞-dim)

V is complete

$\Rightarrow$ $\exists$ basis (e_n) in V such that $\forall F \in V$ $f = \sum f_j e_j$

Theorem: Parseval's Equality

$$\sum f_n F_n \qquad f_n = \frac{(f, F_n)_{L^2}}{\|F_n\|_{L^2}^2}$$

$$\boxed{\sum_{n=0}^{+\infty} \frac{|\langle f, X_n \rangle|^2}{\|X_n\|^2} = \|f\|_{L^2}^2}$$

$$f_N(x) = \sum_{n=0}^{N} \frac{(f, F_n)}{\|F_n\|^2} F_n(x) \longrightarrow f$$

$$\|f - f_N\| \longrightarrow 0 \quad \text{as } N \longrightarrow +\infty$$

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{\frac{in x\pi}{L}}$$

$$f_n = \frac{1}{L} \int_{-L}^{L} f(x) e^{\frac{inx\pi}{L}} dx$$