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Measure Theory & Stochastic Calculus

⓪ Uniform Convergence

A sequence f_n converges uniformly to f in $C[0,1]$ if the sequence $a_n = \sup\{ |f_n(x) - f(x)| : 0 \le x \le 1\}$ converges to 0 as $n \to \infty$.

① Cauchy Sequence

A sequence a_n is a Cauchy sequence if ~~$\forall m,n \geq N$ (for some integer N) we will have~~ for $\epsilon > 0$ we have an integer N such that for all $m,n \geq N$ we have $|a_n - a_m| < \epsilon$. Following can be an example of a Cauchy functional sequence. We define a function $g_n(x)$ such that,

$$g_n(x) = \begin{cases} 0 & , 0 \le x \le \frac{1}{2} \\ n(x - \frac{1}{2}) & , \frac{1}{2} \le x < \frac{1}{2} + \frac{1}{n} \\ 1 & , \text{otherwise} \end{cases}$$

Now we can see that $\int_0^1 |g_n(x) - g_m(x)| < \epsilon$ for $\forall \epsilon > 0$ for a particular integer N and that $m,n \geq N$.

That is $\int_0^1 |g_n(x) - g_m(x)| \to 0$ as $n,m \to \infty$.

Measure

① Null Set

A null set is a set that may be covered by a sequence of intervals of arbitrarily small total length. For example for any small $\varepsilon > 0$ we have,

$$A \subseteq \bigcup_{n=1}^{\infty} I_n$$

and that, $$\sum_{n=1}^{\infty} l(I_n) < \varepsilon$$

$\Rightarrow$ A is 'null'.

① Exercise 2.1 (Capinski)

Any countable set $A = \{x_1, x_2, x_3 \dots\}$ is null.

If A is a countable set consisting of one-element sets, we can cover A in the following way,

$$I_1 = \left(x_1 - \frac{\varepsilon}{8}, x_1 + \frac{\varepsilon}{8}\right) \Rightarrow l(I_n) = \frac{\varepsilon}{2} \cdot \frac{1}{2}$$

$$I_2 = \left(x_2 - \frac{\varepsilon}{16}, x_2 + \frac{\varepsilon}{16}\right) \Rightarrow l(I_2) = \frac{\varepsilon}{2} \cdot \frac{1}{4}$$

$$I_3 = \left(x_3 - \frac{\varepsilon}{32}, x_3 + \frac{\varepsilon}{32}\right) \Rightarrow l(I_3) = \frac{\varepsilon}{2} \cdot \frac{1}{2^3}$$

$\vdots$

$$I_n = \left(x_n - \frac{\varepsilon}{2^{n+2}}, x_n + \frac{\varepsilon}{2^{n+2}}\right) \Rightarrow l(I_n) = \frac{\varepsilon}{2} \cdot \frac{1}{2^n}$$

Therefore $$\sum_{n=1}^{\infty} l(I_n) = \frac{\varepsilon}{2}\left(\frac{1}{2} + \frac{1}{4} + \dots\right) = \frac{\varepsilon}{2} < \varepsilon$$

⊙ Theorem (2.2) – Capinski

If $(N_n)_{n\geq 1}$ is a sequence of null sets, then their union is also null.

$$N = \bigcup_{n=1}^{\infty} N_n$$

We will cover each I_n by intervals. For instance

$$I_1^1 = \left(x_1 - \frac{\varepsilon}{16}, x_1 + \frac{\varepsilon}{16}\right) \Rightarrow l(I_1^1) = \frac{\varepsilon}{4}\cdot\frac{1}{2}$$

$$I_2^1 = \left(x_2 - \frac{\varepsilon}{32}, x_2 + \frac{\varepsilon}{32}\right) \Rightarrow l(I_2^1) = \frac{\varepsilon}{4}\cdot\frac{1}{4}$$

$$\vdots$$

$$I_n^1 = \left(x_n - \frac{\varepsilon}{2^{n+3}}, x_n + \frac{\varepsilon}{2^{n+3}}\right) = \frac{\varepsilon}{4}\cdot\frac{1}{2^n}$$

Hence $l(I^1) = \frac{\varepsilon}{4}\left(\sum \frac{1}{2^n}\right) = \frac{\varepsilon}{4} < \frac{\varepsilon}{2}$

$$\sum_{k=1}^{\infty} l(I_k^1) < \frac{\varepsilon}{2}, \quad N_1 \subseteq \bigcup_{k=1}^{\infty} I_k^1$$

Similarly we can get $\sum_{k=1}^{\infty} l(I_k^2) < \frac{\varepsilon}{4}, \quad N_2 \subseteq \bigcup_{k=1}^{\infty} I_k^2$

$$\vdots$$

$$\sum_{k=1}^{\infty} l(I_k^n) < \frac{\varepsilon}{2^n}, \quad N_n \subseteq \bigcup_{k=1}^{\infty} I_k^n$$

$$N = \bigcup_{n=1}^{\infty} N_n \subseteq \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty} I_k^n$$

Hence $$l\left(\sum_{n,k=1}^{\infty} I_k^n\right) = \sum_{n=1}^{\infty} l\left(\sum_{k=1}^{\infty} I_k^n\right)$$

$$\cancel{\sum_{n=1}^{\infty}} < \sum_{n=1}^{\infty}\left(\frac{\varepsilon}{2^n}\right)$$

$$= \sum_{n=1}^{\infty} \varepsilon\left(\frac{1}{2}+\frac{1}{4}+\dots\right) = \varepsilon$$

○ Ask : Exercise 2.2

○ Ask : Exercise 2.3

○ The (Lebesgue) outer measure of any set $A \subseteq \mathbb{R}$ is given by:

$$m'(A) = \inf Z_A \ , \text{ where } Z_A = \left\{ \sum_{n=1}^{\infty} l(I_n) \ , \ A \subseteq \bigcup_{n=1}^{\infty} I_n \right\}.$$

Clearly we can see that $m'(A) \geq 0$.

Hence the set Z_A is bounded from below by 0. ~~Hence~~ So it is possible that $r \in Z_A \Rightarrow [r, +\infty] \in Z_A$. If $x \in Z_A$ then we can have $[x, \infty] \in Z_A$ and the infimum of Z_A will be $\{x\}$.

~~Hence if A is a null set then $m'(A) = 0$.~~

○ Theorem 2.4 (Capinski) – $A \subseteq \mathbb{R}$ is null iff $m'(A) = 0$

See Page (21) for proof, Capinski.

○ Ask : In Prop 2.5 (Pg 21), if $A \subset B$ then how is $Z_B \subset Z_A$?

○ Theorem 2.6 (Capinski) – The outer measure of any interval equals its length.

(i) First we show that $m'(I) \leq l(I)$.

(ii) Then we show that $l(I) \leq m'(I)$.

We know that $\sum l(I_n) \leq m'(I_n) + \frac{\varepsilon}{2}$. —— (1)

Suppose that I_n is bounded by a_n and b_n. Hence we introduce a J_n such that,

$$J_n = \left(a_n - \frac{\varepsilon}{2^{n+2}}, b_n + \frac{\varepsilon}{2^{n+2}}\right)$$

$$\Rightarrow l(J_n) = l(I_n) \mp \frac{\varepsilon}{2^{n+1}}$$

$$\sum_{n=1}^{\infty} l(I_n) = \sum_{n=1}^{\infty} l(J_n) \mp \frac{\varepsilon}{2}$$

Putting the above in (1) we get,

$$\sum_{n=1}^{\infty} l(J_n) \leq m'(A) + \varepsilon$$

According to Borel Theorem, $A \subseteq \bigcup_{n=1}^{m} (J_n)$

We take $c_n = \inf[a_1, a_2 \ldots a_n]$ and $d_n = \sup[b_1, b_2 \ldots b_n]$

$$l(A) \cancel{\leq l([a,b])} < d_n - c_n < \sum_{n=1}^{m} l(J_n)$$

$$\Rightarrow l(A) < m'(A) + \varepsilon$$

Since we know that $l(A) \geq m'(A)$

$$\Rightarrow l(A) = m'(A)$$

⊙ Theorem 2.7 (Capinski) – Outer (Lebesgue) Measure is countably subadditive. Hence for any sequence of sets $\{E_n\}$ we get,

$$m'\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} m'(E_n)$$

⊙ Prove that $m'(E_1 \cup E_2) \leq m'(E_1) + m'(E_2) + \varepsilon$

⊙ $$\sum_{k=1}^{\infty} l(I_k^1) \leq m(E_1) + \frac{\varepsilon}{2}$$

$$\sum_{k=1}^{\infty} l(I_k^2) \leq m(E_2) + \frac{\varepsilon}{2}$$

$$m'(E_1 \cup E_2) \leq \sum_{k=1}^{\infty} l(I_k^1) + \sum_{k=1}^{\infty} l(I_k^2) \leq m'(E_1) + m'(E_2) + \varepsilon$$

$$\sum_{k=1}^{\infty} l(I_k^n) \leq m'(E_n) + \frac{\varepsilon}{2^n}$$

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} l(I_k^n) \leq \sum_{n=1}^{\infty} m'(E_n) + \varepsilon$$

Since (I_n^k) cover $\bigcup_{n=1}^{\infty} E_n$ we get

$$m'\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n,k=1}^{\infty} l(I_k^n) \leq \sum_{n=1}^{\infty} m'(E_n) + \varepsilon$$

As $\varepsilon \to 0$ we get the desired result that,

$$m'\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} m'(E_n)$$