Lecture 7

Absolute Value and Its Applications

For any real number $a$, we define its **absolute value**, denoted by $|a|$, as follows:

$$|a| = \begin{cases} 
  a, & \text{if } a > 0 \\
  0, & \text{if } a = 0 \\
  -a, & \text{if } a < 0.
\end{cases}$$

Geometrically, any real number $a$ is denoted by a point on the number axis, and the absolute value of $a$ is the distance of the point representing $a$ from the origin of the number axis.

More general, the expression $|a - b|$ denotes the distance between the points on the number axis representing the numbers $a$ and $b$.

When taking absolute value to any algebraic expression, a non-negative value can be obtained always from it by eliminating its negative sign if the value of the expression is “$-$”. This rule is similar to that of taking square to that expression.

**Basic Properties of Absolute Value**

1. $|a| = |-a|$;
2. $-|a| \leq a \leq |a|$;
3. $|a| = |b|$ if and only if $a = b$ or $a = -b$.
4. $|a^n| = |a|^n$ for any positive integer $n$;
5. $|ab| = |a| \cdot |b|$;
6. \[ \left| \frac{a}{b} \right| = \left| \frac{a}{b} \right| \text{ if } b \neq 0; \]
7. \[ |a \pm b| \leq |a| + |b|. \]

**Examples**

**Example 1.** Is there a real number \( x \) such that \( \frac{|x - |x||}{x} \) is a positive number?

**Solution**  
It is clear that \( x \neq 0 \).

For \( x > 0 \), \[ \frac{|x - |x||}{x} = \frac{|x - x|}{x} = \frac{0}{x} = 0. \]

For \( x < 0 \), \[ \frac{|x - |x||}{x} = \frac{|x - (-x)|}{x} = \frac{2x}{x} = \frac{-2x}{x} = -2. \]

Thus, there is no real number \( x \) such that the given fraction is positive.

**Example 2.** If \( a, b, c \) are non-zero real numbers, find all possible values of the expression \( \frac{a}{|a|} + \frac{b}{|b|} + \frac{c}{|c|} \).

**Solution**  
Since \( \frac{x}{|x|} = 1 \) for any \( x > 0 \) and \( \frac{x}{|x|} = -1 \) for any \( x < 0 \),

\[
\frac{a}{|a|} + \frac{b}{|b|} + \frac{c}{|c|} = -3 \text{ if } a, b, c \text{ are all negative};
\]

\[
\frac{a}{|a|} + \frac{b}{|b|} + \frac{c}{|c|} = -1 \text{ if exactly two of } a, b, c \text{ are negative};
\]

\[
\frac{a}{|a|} + \frac{b}{|b|} + \frac{c}{|c|} = 1 \text{ if exactly one of } a, b, c \text{ is negative};
\]

\[
\frac{a}{|a|} + \frac{b}{|b|} + \frac{c}{|c|} = 3 \text{ if } a, b, c \text{ are all positive}.
\]

Thus, the possible values of the given expression are \(-3, -1, 1\) and \(3\).

**Example 3.** Determine the condition for the equality \( \left| \frac{a - b}{a} \right| = \frac{b - a}{a} \).

**Solution**  
The given equality implies that \( a \neq 0 \) and \( \left| \frac{a - b}{a} \right| = -\frac{a - b}{a} \),

therefore \( \frac{a - b}{a} \leq 0 \). Since

\[
\frac{a - b}{a} \leq 0 \iff 1 - \frac{b}{a} \leq 0 \iff \frac{b}{a} \geq 1,
\]
the condition on $a$ and $b$ is $\frac{b}{a} \geq 1$.

**Example 4.** $a, b, c$ are real numbers satisfying $(3a + 6)^2 + \left| \frac{1}{4}b - 10 \right| + |c + 3| = 0$. Find the value of $a^{10} + bc$.

**Solution**  Each term on the left hand side of the given equality is non-negative, we must have

$$3a + 6 = 0 \quad \frac{b}{4} - 10 = 0 \quad c + 3 = 0$$

at the same time, therefore $a = -2, b = 40, c = -3$, so that

$$a^{10} + bc = (-2)^{10} + (40)(-3) = 1024 - 120 = 904.$$  

**Example 5.** Given $1 < x < 3$, simplify the following expressions:

(i) $|x - 3| + |x - 1| - \frac{|x - 1|}{(1 - x)}$

(ii) $|x - 1| + |3 - x|$

**Solution**  For simplifying an expression with absolute values, it is needed to convert it to a normal expression by removing the absolute signs. For this, we need to *partition the range of $x$ into several intervals*, so that on each interval the sign of the expression is fixed (only positive or only negative). For example, for removing the absolute signs of $|x - 3|$, it is needed to take $x - 3 = 0$, i.e. $x = 3$ as the origin, and the sign of $x - 3$ changes at this point: it is positive when $x > 3$, and negative when $x < 3$, so it is needed to discuss $|x - 3|$ for $x > 3$ and $x < 3$ separately. Thus, since the range of $x$ is right to 1 and left to 3,

(i)  $x - 3 < 0$ and $x - 1 > 0$ implies $|x - 3| = -(x - 3), |x - 1| = x - 1$, therefore

$$\frac{|x - 3|}{x - 3} - \frac{|x - 1|}{(1 - x)} = \frac{-(x - 3)}{x - 3} - \frac{x - 1}{1 - x} = -1 - (-1) = 0.$$

(ii) By the same reason, $|x - 1| + |3 - x| = (x - 1) + (3 - x) = 2$.

**Example 6.** Given $1 < x < 3$, simplify $|x - 2| + 2|x|$.

**Solution**  The zero points of $|x - 2|$ and $|x|$ are $x = 2$ and $x = 0$ respectively, it is needed to partition the range of $x$ into two intervals: $1 < x \leq 2$ and $2 < x < 3$.

(i) When $1 < x \leq 2$, $|x - 2| + 2|x| = 2 - x + 2x = 2 + x$

(ii) when $2 < x < 3$, $|x - 2| + 2|x| = x - 2 + 2x = 3x - 2.$

**Example 7.** Simplify $||x + 2| - 7| - |7 - |x - 5||$ for $-2 < x < 5$. 


Solution  We remove the absolute signs from inner layers to outer layers. Since $x + 2 > 0$ and $x - 5 < 0$,
\begin{align*}
|x + 2| - 7 - |7 - |x - 5|| &= |(x + 2) - 7| - |7 - (5 - x)| \\
&= |x - 5| - |2 + x| = (5 - x) - (2 + x) = 3 - 2x.
\end{align*}

Example 8. (AHSME/1990) Determine the number of real solutions of the equation $|x - 2| + |x - 3| = 1$.
(A) 0  (B) 1  (C) 2  (D) 3  (E) more than 3.

Solution  We need to discuss Three cases: $x \leq 2; 2 < x \leq 3$, and $3 < x$.
(i) When $x \leq 2$,
\begin{align*}
|x - 2| + |x - 3| = 1 &\iff (2 - x) + (3 - x) = 1 \iff x = 2;
\end{align*}
(ii) When $2 < x \leq 3$,
\begin{align*}
|x - 2| + |x - 3| = 1 &\iff (x - 2) + (3 - x) = 1 \iff \text{any } x \in (2, 3] \text{ is a solution.}
\end{align*}

Thus, the answer is (E).

Example 9. Let the positions of points on the number axis representing real numbers $a, b, c$ be as shown in the following diagram. Find the value of the expression $|b - a| + |a - c| + |c - b|$.

\begin{figure}[h]
\centering
\begin{tikzpicture}
\draw[thin] (0,0) -- (4,0);
\fill (0,0) circle (2pt); \node at (0,0) {$c$};
\fill (2,0) circle (2pt); \node at (2,0) {$b$};
\fill (4,0) circle (2pt); \node at (4,0) {$a$};
\end{tikzpicture}
\caption{Number axis with points $a, b, c$}
\end{figure}

Solution  From the diagram we find that $c < b < 0 < a < -c$, therefore
\begin{align*}
|b - a| + |a - c| + |c - b| &= (a - b) - (a - c) + b - c = 0.
\end{align*}
Thus, the value of the expression is 0.

Example 10. Given $m = |x + 2| + |x - 1| - |2x - 4|$. Find the maximum value of $m$.

Solution  We discuss the maximum value of $m$ on each of the following three intervals.
(i) When $x \leq -2$, then
\begin{align*}
m &= -(x + 2) - (x - 1) + (2x - 4) = -5.
\end{align*}
(ii) When $-2 < x \leq 1$, then
\begin{align*}
m &= (x + 2) - (x - 1) + (2x - 4) = 2x - 1 \leq 1.
\end{align*}
(iii) When $1 < x \leq 2$, then

$$m = (x + 2) + (x - 1) + (2x - 4) = 4x - 3 \leq 5.$$  

(iv) When $2 < x$, then

$$m = (x + 2) + (x - 1) + (4 - 2x) = 5.$$  

Thus, $m_{\text{max}} = 5.$

**Example 11.** Let $a < b < c$, Find the minimum value of the expression

$$y = |x - a| + |x - b| + |x - c|.$$  

**Solution**

(i) When $x \leq a$,

$$y = (a - x) + (b - x) + (c - x) \geq (b - a) + (c - a).$$

(ii) When $a < x \leq b$,

$$y = (x - a) + (b - x) + (c - x) = (b - a) + (c - x) \geq (b - a) + (c - b) = c - a.$$  

(iii) When $b < x \leq c$,

$$y = (x - a) + (x - b) + (c - x) = (x - a) + (c - b) > (b - a) + (c - b) = c - a.$$  

(iv) When $c < x$,

$$y = (x - a) + (x - b) + (x - c) > (b - a) + (c - b) + (x - c) > c - a.$$  

Thus, $y_{\text{min}} = c - a$, and $y$ reaches this minimum value at $x = b.$

**Testing Questions (A)**

1. Simplify $\frac{|x + |x||}{x}.$

2. Given that $\frac{2x - 1}{3} - 1 \geq x - \frac{5 - 3x}{2}.$ Find the maximum and minimum values of the expression $|x - 1| - |x + 3|.$

3. If real number $x$ satisfies the equation $|1 - x| = 1 + |x|$, then $|x - 1|$ is equal to

   (A) $1$,  \quad (B) $-(x - 1)$,  \quad (C) $x - 1$,  \quad (D) $1 - x$.  


4. What is the minimum value of \(|x + 1| + |x - 2| + |x - 3|)?

5. If \(x < 0\), find the value of \(\frac{|x| - 2x}{3}\).

6. Given \(a < b < c < d\), find the minimum value of \(|x - a| + |x - b| + |x - c| + |x - d|\).

7. If two real numbers \(a\) and \(b\) satisfy \(|a + b| = a - b\), find the value of \(ab\).

8. Given that \(a, b, c\) are integers. If \(|a - b|^{19} + |c - a|^{19} = 1\), find the value of \(|a - c| + |a - b| + |b - c|\).

9. Given \(a = 2009\). Find the value of \(|2a^3 - 3a^2 - 2a + 1| - |2a^3 - 3a^2 - 3a - 2009|\).

10. \(a, b\) are two constants with \(|a| > 0\). If the equation \(||x - a| - b| = 3\) has three distinct solutions for \(x\), find the value of \(b\).

**Testing Questions (B)**

1. Given that \(n\) real numbers \(x_1, x_2, \ldots, x_n\) satisfy \(|x_i| < 1\) (\(i = 1, 2, \ldots, n\)), and
\[|x_1| + |x_2| + \cdots + |x_n| = 49 + |x_1 + x_2 + \cdots + x_n|.
Find the minimum value of \(n\).

2. Given that \(a_1 < a_2 < \cdots < a_n\) are constants, find the minimum value of
\[|x - a_1| + |x - a_2| + \cdots + |x - a_n|.

3. When \(2x + |4 - 5x| + |1 - 3x| + 4\) takes some constant value on some interval, find the interval and the constant value.

4. Given that real numbers \(a, b, c\) are all not zero, and \(a + b + c = 0\). Find the value of \(x^{2007} - 2007x + 2007\), where \(x = \frac{|a|}{b + c} + \frac{|b|}{a + c} - \frac{|c|}{a + b}\).

5. The numbers \(1, 2, 3, \cdots, 199, 200\) are partitioned into two groups of 100 each, and the numbers in one group are arranged in ascending order: \(a_1 < a_2 < a_3 < \cdots < a_{100}\), and those in the other group are arranged in descending order: \(b_1 > b_2 > b_3 > \cdots > b_2 > b_1\). Find the value of the expression
\[|a_1 - b_1| + |a_2 - b_2| + \cdots + |a_{99} - b_{99}| + |a_{100} - b_{100}|.\]
Lecture 8

Linear Equations with Absolute Values

To solve a linear equation with absolute values, we need to remove the absolute value signs in the equation.

In the simplest case $|P(x)| = Q(x)$, where $P(x), Q(x)$ are two expressions with $Q(x) \geq 0$, by the properties of absolute values, we can remove the absolute value signs by using its equivalent form

$$P(x) = Q(x) \quad \text{or} \quad P(x) = -Q(x) \quad \text{or} \quad (P(x))^2 = (Q(x))^2.$$

If there are more than one pair of absolute signs in a same layer, like $|ax + b| - |cx + d| = e$, it is needed to partition the range of the variable $x$ into several intervals to discuss (cf. Lecture 7).

Examples

Example 1. Solve equation $|3x + 2| = 4$.

Solution To remove the absolute signs from $|3x + 2| = 4$ we have

$$|3x + 2| = 4 \iff 3x + 2 = -4 \quad \text{or} \quad 3x + 2 = 4,$$

$$\iff 3x = -6 \quad \text{or} \quad 3x = 2,$$

$$x = -2 \quad \text{or} \quad x = \frac{2}{3}.$$

Example 2. Solve equation $|x - |3x + 1|| = 4$.

Solution For removing multiple layers of absolute value signs, we remove them layer by layer from outer layer to inner layer. From the given equation we have $x - |3x + 1| = 4$ or $x - |3x + 1| = -4$.

From the first equation $x - |3x + 1| = 4$ we have

$$x - |3x + 1| = 4 \iff 0 \leq x - 4 = |3x + 1| \iff -x + 4 = 3x + 1 \text{ or } x - 4 = 3x + 1,$$
therefore $x = \frac{3}{4}$ or $x = \frac{5}{2}$, which contradict the requirement $x \geq 4$, so the two solutions are not acceptable.

The second equation $x - |3x + 1| = -4$ implies that

$$x - |3x + 1| = -4 \iff 0 \leq x + 4 = |3x + 1| \iff -x - 4 = 3x + 1 \text{ or } x + 4 = 3x + 1,$$

therefore $x_1 = -\frac{5}{4}$, $x_2 = \frac{3}{2}$.

**Example 3.** Solve equation $|||x| - 2| - 1| = 3$.

**Solution** There are three layers of absolute values. Similar to Example 2,

$|||x| - 2| - 1| = 3 \iff ||x| - 2| - 1 = 3 \text{ or } ||x| - 2| - 1 = -3$

$\iff |x| - 2 = 4 \text{ or } ||x| - 2| - 1 = -2$ (no solution)

$\iff |x| - 2 = 4 \text{ or } |x| - 2 = -4 \iff |x| = 6 \text{ or } |x| = -2$ (no solution)

$\iff |x| = 6 \iff x_1 = 6, \ x_2 = -6$.

**Example 4.** If $|x - 2| + x - 2 = 0$, then the range of $x$ is

(A) $x > 2$, (B) $x < 2$, (C) $x \geq 2$, (D) $x \leq 2$.

**Solution** The given equation produces $|x - 2| = 2 - x$, so $x \leq 2$ and

$|x - 2| = 2 - x \iff x - 2 = 2 - x \text{ or } x - 2 = -x \iff x = 2 \text{ or } x \leq 2$.

Since $x = 2$ is contained by the set $x \leq 2$, the answer is $x \leq 2$, i.e. (D).

**Example 5.** If $||4m + 5| - b| = 6$ is an equation in $m$, and it has three distinct solutions, find the value of the rational number $b$.

**Solution** From the given equation we have (i) $|4m + 5| - b = 6$ or (ii) $|4m + 5| - b = -6$.

If (i) has exactly one solution, then $b + 6 = 0$, i.e. $b = -6$ which implies (ii) should be $|4m + 5| = -12$, so no solutions. Thus $b \neq -6$ and (i) has two solutions but (ii) has exactly one solution, so $b - 6 = 0$, i.e. $b = 6$. In fact, when $b = 6$ then (i) becomes $|4m + 5| = 12$,

$$4m + 5 = 12 \quad \text{or} \quad 4m + 5 = -12,$$

$m = \frac{7}{4}$ or $m = -\frac{17}{4}$,

and, from (ii) the third root $m = -\frac{5}{4}$.

**Example 6.** Solve equation $|x - 1| + 2|x| - 3|x + 1| - |x + 2| = x$. 


Solution  Letting each of \(|x - 1|, |x|, |x + 1|, |x + 2|\) be 0, we get \(x = 1, 0, -1, -2\). By using these four points as partition points, the number axis is partitioned as five intervals: \(x \leq -2, -2 < x \leq -1, -1 < x \leq 0, 0 < x \leq 1, 1 < x\).

(i) When \(x \leq -2\), then
\[
(1 - x) + 2(-x) + 3(x + 1) + (x + 2) = x \iff 6 = 0, \quad \therefore \text{no solution;}
\]

(ii) when \(-2 < x \leq -1\), then
\[
(1 - x) + 2(-x) + 3(x + 1) - (x + 2) = x, \iff x = 1 \quad \therefore \text{no solution;}
\]

(iii) when \(-1 < x \leq 0\), then
\[
1 - x + 2(-x) - 3(x + 1) - (x + 2) = x, \iff 8x = -4, \quad \therefore x = -\frac{1}{2};
\]

(iv) when \(0 < x \leq 1\), then
\[
(1 - x) + 2x - 3(x + 1) - (x + 2) = x \iff 4x = -4, \quad \therefore x = -1,
\]
therefore no solution;

(v) when \(1 < x\), then
\[
(x - 1) + 2x - 3(x + 1) - (x + 2) = x \iff 2x = -6 \quad x = -3
\]
therefore no solution.

Thus \(x = -\frac{1}{2}\) is the unique solution.

Example 7. If \(|x + 1| + (y + 2)^2 = 0\) and \(ax - 3ay = 1\), find the value of \(a\).

Solution  Since \(|x + 1| \geq 0\) and \((y + 2)^2 \geq 0\) for any real \(x, y\), so \(x + 1 = 0\) and \(y + 2 = 0\), i.e. \(x = -1, y = -2\). By substituting them into \(ax - 3ay = 1\), it follows that \(-a + 6a = 1\), therefore \(a = \frac{1}{5}\).

Example 8. How many pairs \((x, y)\) of two integers satisfy the equation \(|xy| + |x - y| = 1\)?

Solution  \(|xy| \geq 0\) and \(|x - y| \geq 0\) implies that

(i) \(|xy| = 1, |x - y| = 0\) or  (ii) \(|xy| = 0, |x - y| = 1\).

(i) implies that \(x = y\) and \(x^2 = y^2 = 1\), so its solutions \((x, y)\) are \((1, 1)\) or \((-1, -1)\).

(ii) implies that at least one of \(x, y\) is 0. When \(x = 0\), then \(|y| = 1\), i.e. \(y = \pm 1\); if \(y = 0\), then \(|x| = 1\), i.e. \(x = \pm 1\). Hence the four solutions for \((x, y)\) are \((0, 1), (0, -1), ((1, 0), (-1, 0))\).

Thus there are 6 solutions for \((x, y)\) in total.

Example 9. If \(|x + 1| - |x - 3| = a\) is an equation in \(x\), and it has infinitely many solutions, find the value of \(a\).
Solution  By $-1, 3$ partition the number axis into three parts: $x \leq -1, \ -1 < x \leq 3, \ 3 < x$.

(i) When $x \leq -1$, then
\[-(x+1)-(3-x) = a \iff -4 = a.\] Therefore any value of $x$ not greater than $-1$ is a solution if $a = -4$.

(ii) When $-1 < x \leq 3$, then
\[(x+1) - (3-x) = a \iff 2x = a + 2 \iff x = \frac{1}{2}(a + 2),\] i.e. the solution is unique if any.

(iii) When $3 < x$, then
\[(x+1) - (x-3) = a \iff 4 = a.\] Therefore any value of $x$ greater than $3$ is a solution if $a = 4$.

Thus, the possible values of $a$ are $-4$ and $4$.

Example 10. (AHSME/1988) If $|x| + x + y = 10$ and $x + |y| - y = 12$, find $x + y$.

Solution  There are four possible cases: (i) $x, y > 0$; (ii) $x, y \leq 0$; (iii) $x > 0, y \leq 0$ and (iv) $x \leq 0, y > 0$.

(i) If $x > 0, y > 0$ then $2x + y = 10, x = 12 \iff y < 0$, a contradiction, so no solution;

(ii) If $x \leq 0$ and $y \leq 0$, then $y = 10$, a contradiction, so no solution;

(iii) If $x > 0, y \leq 0$, then $2x + y = 10, x - 2y = 12$. By eliminating $y$, we have $x = \frac{32}{5}$, so $y = -\frac{14}{5}$.

Thus, $x + y = \frac{18}{5}$.

Testing Questions  (A)

1. Solve equation $|5x - 4| - 2x = 3$.

2. (CHINA/2000) $a$ is an integer satisfying the equation $|2a + 7| + |2a - 1| = 8$. Then the number of solutions for $a$ is
   (A) 5   (B) 4   (C) 3   (D) 2.
3. (AHSME/1984) The number of distinct solutions of the equation \(|x - |2x + 1|| = 3\) is
   (A) 0,  (B) 1,  (C) 2,  (D) 3,  (E) 4.

4. (CHNMOL/1987) Given that the equation \(|x| = ax + 1\) has exactly one negative solution and has no positive solution. then the range of \(a\) is
   (A) \(a > 1\),  (B) \(a = 1\),  (C) \(a \geq 1\),  (D) none of preceding.

5. (CHNMOL/1986) If the equation \(||x - 2| - 1|| = a\) has exactly three integer solution for \(x\), then the value of \(a\) is
   (A) 0,  (B) 1,  (C) 2,  (D) 3.

6. If the equation \(\frac{a}{2008} |x| - x - 2008 = 0\) has only negative solutions for \(x\), find the range of \(a\).

7. In the equations in \(x\)
   (i) \(|3x - 4| + 2m = 0\),  (ii) \(|4x - 5| + 3n = 0\),  (iii) \(|5x - 6| + 4k = 0\),
   \(m, n, k\) are constants such that (i) has no solution, (ii) has exactly one solution, and (iii) has two solutions. Then
   (A) \(m > n > k\),  (B) \(n > k > m\),  (C) \(k > m > n\),  (D) \(m > k > n\).

8. Solve the system of simultaneous equations
   \[
   \begin{cases}
   |x - y| = x + y - 2, \\
   |x + y| = x + 2.
   \end{cases}
   \]

9. (AHSME/1958) We may say concerning the solution of \(|x|^2 + |x| - 6 = 0\) that:
   (A) there is only one root;  (B) the sum of the roots is 1;
   (C) the sum of the roots is 0;  (D) the product of the roots is +4;
   (E) the product of the roots is −6.

10. (CHINA/2001) Solve the system \(x + 3y + |3x - y| = 19, 2x + y = 6\).

**Testing Questions  (B)**

1. Solve the system
   \[
   \begin{cases}
   |x - 2y| = 1, \\
   |x| + |y| = 2.
   \end{cases}
   \]
2. (CHINA/1990) for the equation with 4 layers of absolute value signs \[ |x - 1| - 1| - 1| - 1| = 0, \]
   (A) the solutions are 0, 2, 4;    (B) 0, 2, 4 are not solutions;
   (C) the solutions are within the three values 0, 2, 4;
   (D) 0, 2, 4 are not all of the solutions.

3. (CHNMOL/1995) Given that \(a, b\) are real numbers satisfying the inequality
   \[ |a| - (a + b) < |a - |a + b||, \]
   (A) \(a > 0, b > 0\)    (B) \(a < 0, b > 0\)    (C) \(a > 0, b < 0\)    (D) \(a < 0, b < 0\).

4. Given that \(\frac{1}{|x - 2|} = \frac{1}{|x - 52a|}\) is an equation in \(x\),
   (i) solve the equation,    (ii) prove that the solutions must be composite numbers if \(a\) is the square of an odd prime number.

5. (IMO/1966) Solve the system of equations
   \[
   \begin{align*}
   |a_2 - a_1| x_1 & + |a_1 - a_2| x_2 + |a_1 - a_3| x_3 + |a_1 - a_4| x_4 = 1 \\
   |a_3 - a_1| x_1 & + |a_3 - a_2| x_2 + |a_2 - a_3| x_3 + |a_2 - a_4| x_4 = 1 \\
   |a_4 - a_1| x_1 & + |a_4 - a_2| x_2 + |a_3 - a_4| x_3 + |a_3 - a_4| x_4 = 1
   \end{align*}
   \]
   where \(a_1, a_2, a_3, a_4\) are four different real numbers.