Martingales

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$$E[Z_n | Z_m, Z_{m-1}, \ldots, Z_1] = Z_n \quad (\text{given } m > n)$$

Examples,

1) $Z_n = \sum_{i=1}^{n} X_i$ (X_i are iid and zero-mean) $\Rightarrow \{Z_n; n \geq 1\}$ is a martingale.

2) $Z_n = \sum X_i \qquad E[X_i | X_{i-1} \ldots X_1] = 0$ for each $i \geq 1$

$\{Z_n; n \geq\}$ is a martingale.

3) $X_i = U_i Y_i \qquad \{U_i; i \geq 1\}$ are IID and equiprobable ± 1.

Then $\{Z_n; n \geq 1\}$, where $Z_n = X_1 + X_2 + \cdots + X_n$.

4) $\{X_i; i \geq 1\}$ be a sequence of iid rv's unit-mean

Then $\{Z_n; n \geq 1\}$, where $Z_n = X_1 X_2 \ldots X_n$, is a martingale.

$$P_X(0) = P_X(2) = \frac{1}{2}$$

$$\Rightarrow \quad P_{Z_n}(2^n) = 2^{-n}$$

$$P_{Z_n}(0) = 1 - 2^{-n}$$

Hence $\lim_{n \to \infty} Z_n = 0$ wp 1

$$\lim_{n \to \infty} E(Z_n) = 1$$

Sub- and Super-martingales

$\{Z_n ; n \geq 1\}$

Submartingales grow in time $\Longrightarrow E[Z_n | Z_{n-1}, Z_{n-2}, \ldots Z_1] \geq Z_{n-1}$

Supermartingales shrink in time $\Longrightarrow E[Z_n | Z_{n-1}, \ldots Z_1] \leq Z_{n-1}$

If Z_n is a submartingale, then $-Z_n$ is a supermartingale.

For submartingales,

$$E[Z_n | Z_i \ldots Z_1] \geq Z_i \ , \quad \text{for all } n > i \geq 1$$

$$\text{and } E[Z_n] \geq E[Z_i] \ , \quad \text{for all } n > i \geq 0.$$

Convex Functions

$h : \mathbb{R} \to \mathbb{R}$ is convex if each tangent lies on or below the curve.

Jensen Inequality: If h is convex and Z is a rv with finite expectation, then

$$h(E[Z]) \leq E[h(Z)]$$

Jensen inequality leads to the following theorem:

$\Rightarrow$ If $\{Z_n ; n \geq 1\}$ is a martingale or submartingale, if h is convex and if $E[|h(Z_n)|] < \infty$ for all n, then $\{h(Z_n) ; n \geq 1]$ is a submartingale.

$\Rightarrow$ If $\{Z_n ; n \geq 1\}$ is a martingale, then $\{|Z_n| ; n \geq 1\}$ $\{Z_n^2 ; n \geq 1\}$ and $\{e^{rZ_n} ; n \geq 1\}$ are submartingales.

Stopped Martingales

The definition of a stopping trial J for a stochastic process $\{Z_n; n \geq 1\}$ applies to any process. That is, J must be a rv and $\{J=n\}$ must be specified by $\{Z_1, Z_2 \dots Z_n\}$.

A possibly defective stopping trial is thus a stopping rule in which stopping may never happen.
A Stopped process $\{Z_n^*; n \geq 1\}$ for a possibly defective stopping time J on the process $\{Z_n; n \geq 1\}$ satisfies $Z_n^* = Z_n$ if $n \leq J$ and $Z_n^* = Z_J$ if $n > J$.

Theorem: If J is a possibly defective stopping ~~time~~ rule for a martingale $\{Z_n; n \geq 1\}$ then the stopped process $\{Z_n^*; n \geq 1\}$ is a martingale. So is the case for submartingale.

Proof: You stop at m.

$$Z_n^* = \sum_{m=1}^{n-1} Z_m \mathbb{I}_{\{J=m\}} + Z_n \mathbb{I}_{\{J \geq n\}} \quad \text{(All are zero's except 1).}$$

$$|Z_n^*| \leq \sum_{m<n} |Z_m| + |Z_n| \quad \text{Thus, } E[|Z_n^*|] < \infty$$

Since it is bounded by sum of n finite numbers.

Let $\vec{Z}^{(n-1)}$ denote $Z_{n-1}, Z_{n-2}, \dots, Z_1$ and consider

$$E[Z_n^* | \vec{Z}^{(n-1)}] = \sum_{m<n} E[Z_m \mathbb{I}_{\{J=m\}} | \vec{Z}^{(n-1)}] + E[Z_n \mathbb{I}_{\{J \geq n\}} | \vec{Z}^{(n-1)}]$$

$$E\left[Z_m \mathbb{I}_{\{J=m\}} \mid \vec{Z}^{(n-1)}\right] = Z_m \mathbb{I}_{\{J=m\}} \quad (E[\dots]=0,\ J \neq m)$$

$$E\left[Z_n \mathbb{I}_{\{J \geq n\}} \mid \vec{Z}^{(n-1)}\right] = Z_{n-1} \mathbb{I}_{\{J \geq n\}}$$

$$E\left[Z_n^{*} \mid \vec{Z}^{(n-1)}\right] = \sum_{m<n} Z_m \mathbb{I}_{\{J=m\}} + Z_{n-1} \mathbb{I}_{\{J \geq n\}}$$

$$= \sum_{m<n-1} Z_m \mathbb{I}_{\{J=m\}} + Z_{n-1}\left[\mathbb{I}_{\{J=n-1\}} + \mathbb{I}_{\{J \geq n\}}\right] \quad (\dagger)$$

$$= Z_{n-1}^{*}$$

($\dagger$) Because in the case when $J=n-1$, we have $\sum_{m<n} Z_m \mathbb{I}_{\{J=m\}} = Z_{n-1}$.

This shows that $E[Z_n^{*} \mid \vec{Z}^{(n-1)}] = Z_{n-1}^{*}$. However to prove that $\{Z_n^{*}, n \geq 1\}$ is a martingale, we must show that $E[Z_n^{*} \mid \vec{Z}^{*(n-1)}] = Z_{n-1}^{*}$.

However $\vec{Z}^{*(n-1)}$ is a function of $\vec{Z}^{(n-1)}$.

For every sample point $\vec{z}^{(n-1)}$ of $\vec{Z}^{(n-1)}$ leading to a given $s^{*(n-1)}$ of $\vec{Z}^{*(n-1)}$, we have

$$E\left[Z_n^{*} \mid \vec{Z}^{(n-1)} = \vec{z}^{(n-1)}\right] = s^{*(n-1)}$$

$$E\left[Z_n^{*} \mid \vec{Z}^{*(n-1)} = \vec{z}^{*(n-1)}\right] = \vec{z}^{*(n-1)}$$

Hence $\{Z_n^{*}; n \geq 1\}$ is a martingale since $E\left[Z_n^{*} \mid \vec{Z}^{*(n-1)}\right] = Z_{n-1}^{*}$

Consequently, $E[Z_1] \leq E[Z_n^{*}] \leq E[Z_n]$ (Submartingale)

$E[Z_1] = E[Z_n^{*}] = E[Z_n]$ (Martingale)

Kolmogorov Submartingale Inequality

Theorem: Let $\{Z_n; n \geq 1\}$ be a non-negative submartingale. Then for any positive integer m, and any $a > 0$,

$$P\left\{ \max_{1 \leq i \leq m} Z_i \geq a \right\} \leq \frac{E[Z_m]}{a}$$

Proof: Let J be the stopping time that is essentially the smallest $n \leq m$ such that $Z_n \geq a$. More specifically if $Z_n \geq a$ for some $n \leq m$, then J is the smallest n for which $Z_n \geq a$.

If $Z_n < a$ for all $n \leq m$, then $J = m$. Thus the process must stop by time m, and $Z_J \geq a$ iff $Z_n \geq a$ for some $n \leq m$. Thus,

$$P\left\{ \max_{1 \leq n \leq m} Z_n \geq a \right\} = P\{ Z_J \geq a \} \leq \frac{E[Z_J]}{a}$$

(By Markov).

Since the process must be stopped by time m, we have $Z_J = Z_m^*$.

$E[Z_m^*] \leq E[Z_m]$ so we get $\dfrac{E[Z_J]}{a} \leq \dfrac{E[Z_m]}{a}$, completing the proof. $\square$

We see that Kol. Sub. Inq. is a generalization of the Markov inequality and in the similar manner we can state that,

Let $\{Z_n; n \geq \}$ be a martingale with $E[Z_n^2] < \infty$ for all $n \geq 1$.

Then $P\left\{ \max_{1 \leq n \leq m} |Z_n| \geq b \right\} \leq \dfrac{E[Z_m^2]}{b^2}$, for all $m \geq 2$, $b > 0$.

And Let S_n be a RW ($S_n = \sum X_i$) with each X_i has mean $\overline{X}$ and variance σ^2.

Then for $m \in \mathbb{Z}^+$, $\epsilon > 0$

$$P\left\{ \max_{1 \leq n \leq m} |S_n - n\overline{X}| \geq m\epsilon \right\} \leq \frac{\sigma^2}{m\epsilon^2}$$

Martingale Convergence Theorem

Let $\{X_n\}$ be a submartingale. Suppose that $\sup_n E|X_n| < \infty$. Then there is a finite random variable X such that $X_n \to X$ almost surely.

We use the Upcrossing Lemma.

Let $\{X_n\}$ be a sub martingale. For $M \in R$ and $\alpha < \beta$, let

$$U_M^{\alpha,\beta} = \sup\{k ; a_1 < b_1 < a_2 < b_2 < \dots < a_k < b_k < M ;$$
$$X_{a_i} \le \alpha, X_{b_i} \ge \beta\}$$

Then $E[U_M^{\alpha,\beta}] \le \dfrac{E[|X_M - X_0|]}{\beta - \alpha}$

Proof. If $\alpha \in R$ and $\{X_n\}$ a submartingale, then it can be observed that $\{\max(X_n,\alpha)\}$ is also a submartingale, and it will have the same number of upcrossings of $[\alpha, \beta]$ as $\{X_n\}$.

We can see that $|\max(X_M,\alpha) - \max(X_0,\alpha)| \le |X_M - X_0|$. Hence for the remaining part of the proof we replace X_n by $\max(X_n,\alpha)$.

We assume that $X_n \ge \alpha$ for all n.

Let $u_0 = v_0 = 0$ and iteratively define u_j and v_j for $j \ge 1$ by

$$u_j = \min(M, \inf\{k > v_{j-1} ; X_k \le \alpha\})$$

$$v_j = \min(M, \inf\{k > u_j ; X_k \ge \beta\})$$

We will necessarily have $v_m = M$ so that $E(X_M) = E(X_{v_M})$.

$$= E(X_{V_m} - X_{U_m} + X_{U_m} - X_{V_{m-1}} + X_{V_{m-1}} + \dots + X_{U_1} - X_0 + X_0)$$

$$= E(X_0) + E\left(\sum_{k=1}^{m}(X_{V_k} - X_{U_k})\right) + \sum_{k=1}^{m} E\left[X_{U_k} - X_{V_{k-1}}\right] \qquad (\star\star)$$

We know that $E(X_{U_k} - X_{V_{k-1}}) \geq 0$ (Stopping Time Theorem), since $\{X\}$ is a submartingale and $V_{k-1} \leq U_k \leq M$. Hence $\sum_{k=1}^{M} E(X_{U_k} - X_{V_{k-1}}) \geq 0$.

($\star$) This is counter-intuitive because $X_{U_k} \leq \alpha$ and $X_{V_{k-1}} \geq \beta$. However we are talking about the expectation and not what has happened!

We claim that $$E\left(\sum_{k=1}^{m}(X_{V_k} - X_{U_k})\right) \geq E\left((\beta-\alpha)U_M^{\alpha,\beta}\right)$$

Indeed each upcrossing contributes at least $\beta - \alpha$ to the sum. And any "null cycle" where $U_k = V_k = M$ contributes nothing since $X_{V_k} - X_{U_k} = X_M - X_M = 0$.

Finally we may have "incomplete cycle" where $U_k < M$ but $V_k = M$. However we are assuming that $X_n \geq \alpha$ we must have

$X_{V_k} - X_{U_k} = X_M - X_{U_k} \geq \alpha - \alpha = 0$, that is such an incomplete cycle can only increase the sum.

Hence $$E\left(\sum_{k=1}^{M}(X_{V_k} - X_{U_k})\right) \geq (\beta-\alpha)E(U_M^{\alpha,\beta}).$$

From ($\star\star$) we conclude that $E(X_M) \geq E(X_0) + (\beta-\alpha)E(U_M^{\alpha,\beta})$

By linearity of expectation this is another way to write the ~~theorem~~ Lemma.

Proof of the Martingale Convergence Theorem

Let $K = \sup_n E(|X_n|) < \infty$, Note that by Fatou's Lemma,

$$E(\liminf_{n\to\infty} |X_n|) \le \liminf_{n\to\infty} E(|X_n|) \le K < \infty$$

This means that $P(|X_n| \to \infty) = 0$.

We suppose that $P(\liminf X_n < \limsup X_n) > 0$. (We have to prove it wrong).

$\{\liminf X_n < \limsup X_n\}$

$$= \bigcup_{q\in\mathbb{Q}} \bigcup_{k\in\mathbb{N}} \left\{\liminf X_n < q < q + \frac{1}{k} < \limsup X_n\right\}$$

From countable subadditivity we can find $\alpha, \beta \in \mathbb{Q}$ with

$$P(\liminf X_n < \alpha < \beta < \limsup X_n) > 0$$

With $U_M^{\alpha,\beta}$ as in Upcrossing Lemma this means $P(\lim_{M\to\infty} U_M^{\alpha,\beta} = \infty) > 0$.

so $E(\lim_{M\to\infty} U_M^{\alpha,\beta}) = \infty$. Explanation: As time becomes indefinite, the upcrossings will be infinite with probability greater than zero, and the expected number of upcrossings will be infinite.

Now by the MCT, $\lim_{M\to\infty} E(U_M^{\alpha,\beta}) = E(\lim_{M\to\infty} U_M^{\alpha,\beta}) = \infty$

But the Upcrossing Lemma says that for $M \in \mathbb{N}$,

$$E(U_M^{\alpha,\beta}) \le \frac{E|X_M - X_0|}{\beta - \alpha} \le \frac{2K}{\beta-\alpha}$$ & ($2K$ because $K = \sup_n E(|X_n|)$).

Hence $P(\liminf_n |X_n| = \infty) = 0$ and $P(\liminf X_n < \limsup X_n) = 0$

Hence $\lim X_n$ exists and is finite!