NYUAD Mathematics Club

Problem-Solving Session

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International Mathematics Competition

IMC

Organized By:

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MATHEMATICS CLUB

ABU DHABI

The Pigeonhole Principle

Problem 1.1

Let p be a prime number and a, b, c integers such that a and b are not divisible by p. Prove that the equation $ax^2 + by^2 \equiv c \pmod{p}$ has integer solutions.

Problem 1.2

In each of the unit squares of a 10×10 checkerboard, a positive integer not exceeding 10 is written. Any two numbers that appear in adjacent or diagonally adjacent squares of the board are relatively prime. Prove that some number appears at least 17 times.

Problem 1.3

Let $x_1 = x_2 = x_3 = 1$ and $x_{n+3} = x_n + x_{n+1}x_{n+2}$ for all positive integers n. Prove that for any positive integer m there is an index k such that m divides x_k.

Ordered Sets and Extremal Elements

Problem 2.1

Let $a_1, a_2, \ldots, a_n, \ldots$ be a sequence of distinct positive integers. Prove that for any positive integer n,

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq \frac{2n+1}{3}(a_1 + a_2 + \cdots + a_n).$$

Problem 2.2

Let X be a subset of the positive integers with the property that the sum of any two not necessarily distinct elements in X is again in X. Suppose that $\{a_1, a_2, \ldots, a_n\}$ is the set of all positive integers not in X. Prove that $a_1 + a_2 + \cdots + a_n \leq n^2$.

Problem 2.3

At a party assume that no boy dances with all the girls, but each girl dances with at least one boy. Prove that there are two girl–boy couples gb and $g'b'$ who dance, whereas b does not dance with g', and g does not dance with b'.

Invariants and Semi-Invariants

Problem 3.1

Several positive integers are written on a blackboard. One can erase any two distinct integers and write their greatest common divisor and least common multiple instead. Prove that eventually the numbers will stop changing.

Problem 3.2

There is a heap of 1001 stones on a table. You are allowed to perform the following operation: you choose one of the heaps containing more than one stone, throw away a stone from the heap, then divide it into two smaller (not necessarily equal) heaps. Is it possible to reach a situation in which all the heaps on the table contain exactly 3 stones by performing the operation finitely many times?

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