

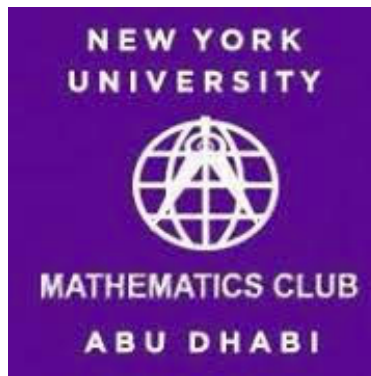
NYUAD Mathematics Club

Problem-Solving Session

26-Sept-2015, E-048



Organized By:



Mathematical Induction

Problem 1.1

Prove for all positive integers n the identity

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2n-1} - \frac{1}{2n}.$$

Problem 1.2

Let n be a positive integer. Prove that

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3} < \frac{3}{2}.$$

Proof by Contradiction

Problem 2.1

Determine all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ satisfying

$$xf(y) + yf(x) = (x+y)f(x^2 + y^2)$$

for all positive integers x and y .

Problem 2.2

Let $n > 1$ be an arbitrary real number and let k be the number of positive prime numbers less than or equal to n . Select $k + 1$ positive integers such that none of them divides the product of all the others. Prove that there exists a number among the chosen $k + 1$ that is bigger than n .

The Cauchy-Schwarz Inequality

Problem 3.1

Prove that

$$\frac{\sin^3 a}{\sin b} + \frac{\cos^3 a}{\cos b} \geq \sec(a - b),$$

for all $a, b \in (0, \frac{\pi}{2})$.

Problem 3.2

Prove that

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{2\sqrt[3]{abc}} \geq \frac{(a+b+c + \sqrt[3]{abc})^2}{(a+b)(b+c)(c+a)},$$

for all $a, b, c > 0$.

The Triangle Inequality

Problem 4.1

For positive numbers a, b, c prove the inequality

$$\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \geq \sqrt{a^2 + ac + c^2}.$$

Problem 4.2

Given the vectors $\vec{a}, \vec{b}, \vec{c}$ in the plane, show that

$$\|\vec{a}\| + \|\vec{b}\| + \|\vec{c}\| + \|\vec{a} + \vec{b} + \vec{c}\| \geq \|\vec{a} + \vec{b}\| + \|\vec{a} + \vec{c}\| + \|\vec{b} + \vec{c}\|.$$

Strum's Principle

Problem 5.1

Let $x_1, x_2, \dots, x_n, n \geq 2$, be positive numbers such that $x_1 + x_2 + \dots + x_n = 1$. Prove that

$$\left(1 + \frac{1}{x_1}\right) \left(1 + \frac{1}{x_2}\right) \cdots \left(1 + \frac{1}{x_n}\right) \geq (n+1)^n.$$

Problem 5.2

Let $a, b, c > 0, a + b + c = 1$. Prove that

$$0 \leq ab + bc + ac - 2abc \leq \frac{7}{27}.$$

Problem 5.3

Let x_1, x_2, \dots, x_n be n real numbers such that $0 < x_j \leq \frac{1}{2}$, for $1 \leq j \leq n$. Prove the inequality

$$\frac{\prod_{j=1}^n x_j}{\left(\sum_{j=1}^n x_j\right)^n} \leq \frac{\prod_{j=1}^n (1 - x_j)}{\left(\sum_{j=1}^n (1 - x_j)\right)^n}.$$

Viète's Relations

Problem 6.1

Prove that there are unique positive integers a, n such that

$$a^{n+1} - (a + 1)^n = 2001.$$

Irreducible Polynomials

Problem 7.1

Prove that for any distinct integers a_1, a_2, \dots, a_n the polynomial

$$P(x) = (x - a_1)^2(x - a_2)^2 \cdots (x - a_n)^2 + 1$$

cannot be written as a product of two nonconstant polynomials with integer coefficients.

Problem 7.2

Let p be a prime number. Prove that the polynomial

$$P(x) = x^{p-1} + 2x^{p-2} + 3x^{p-3} + \cdots + (p-1)x + p$$

is irreducible in $\mathbb{Z}[x]$.
