NYUAD Mathematics Club

Problem-Solving Session

26-Sept-2015, E-048

International Mathematics Competition

IMC

Organized By:

NEW YORK UNIVERSITY

MATHEMATICS CLUB

ABU DHABI

Mathematical Induction

Problem 1.1

Prove for all positive integers n the identity

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2n-1} - \frac{1}{2n}.$$

Problem 1.2

Let n be a positive integer. Prove that

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3} < \frac{3}{2}.$$

Proof by Contradiction

Problem 2.1

Determine all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$xf(y) + yf(x) = (x+y)f(x^2+y^2)$$

for all positive integers x and y.

Problem 2.2

Let $n > 1$ be an arbitrary real number and let k be the number of positive prime numbers less than or equal to n. Select $k+1$ positive integers such that none of them divides the product of all the others. Prove that there exists a number among the chosen $k+1$ that is bigger than n.

The Cauchy-Schwarz Inequality

Problem 3.1

Prove that

$$\frac{\sin^3 a}{\sin b} + \frac{\cos^3 a}{\cos b} \geq \sec(a-b),$$

for all $a, b \in (0, \frac{\pi}{2})$.

Problem 3.2

Prove that

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{2\sqrt[3]{abc}} \geq \frac{(a+b+c+\sqrt[3]{abc})^2}{(a+b)(b+c)(c+a)},$$

for all $a, b, c > 0$.

The Triangle Inequality

Problem 4.1

For positive numbers a, b, c prove the inequality

$$\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2} \geq \sqrt{a^2+ac+c^2}.$$

Problem 4.2

Given the vectors $\vec{a}, \vec{b}, \vec{c}$ in the plane, show that

$$\|\vec{a}\|+\|\vec{b}\|+\|\vec{c}\|+\|\vec{a}+\vec{b}+\vec{c}\| \geq \|\vec{a}+\vec{b}\|+\|\vec{a}+\vec{c}\|+\|\vec{b}+\vec{c}\|.$$

Strum's Principle

Problem 5.1

Let $x_1, x_2, \ldots, x_n$, $n \geq 2$, be positive numbers such that $x_1+x_2+\cdots+x_n = 1$. Prove that

$$\left(1+\frac{1}{x_1}\right)\left(1+\frac{1}{x_2}\right)\cdots\left(1+\frac{1}{x_n}\right) \geq (n+1)^n.$$

Problem 5.2

Let $a, b, c > 0$, $a+b+c = 1$. Prove that

$$0 \leq ab+bc+ac-2abc \leq \frac{7}{27}.$$

Problem 5.3

Let $x_1, x_2, \ldots, x_n$ be n real numbers such that $0 < x_j \leq \frac{1}{2}$, for $1 \leq j \leq n$. Prove the inequality

$$\frac{\prod_{j=1}^{n} x_j}{\left(\sum_{j=1}^{n} x_j\right)^n} \leq \frac{\prod_{j=1}^{n}(1-x_j)}{\left(\sum_{j=1}^{n}(1-x_j)\right)^n}.$$

Viete's Relations

Problem 6.1

Prove that there are unique positive integers a, n such that

$$a^{n+1} - (a+1)^n = 2001.$$

Irreducible Polynomials

Problem 7.1

Prove that for any distinct integers $a_1, a_2, \ldots, a_n$ the polynomial

$$P(x) = (x-a_1)^2(x-a_2)^2 \cdots (x-a_n)^2 + 1$$

cannot be written as a product of two nonconstant polynomials with integer coefficients.

Problem 7.2

Let p be a prime number. Prove that the polynomial

$$P(x) = x^{p-1} + 2x^{p-2} + 3x^{p-3} + \cdots + (p-1)x + p$$

is irreducible in $\mathbb{Z}[x]$.

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