NYUAD Mathematics Club

Problem-Solving Session

30-Oct-2015, E-048

Organized By:

Operations With Matrices

Problem 1.1

Do there exist $n \times n$ matrices A and B such that $AB - BA = \mathcal{I}_n$?

Problem 1.2

Let A and B be two $n \times n$ matrices that do not commute and for which there exist nonzero real numbers p, q, r such that $pAB + qBA = \mathcal{I}_n$ and $A^2 = rB^2$. Prove that $p = q$.

Transposes and Hermitian Conjugates

Problem 2.1

Let M be an $n \times n$ complex matrix. Prove that there exist Hermitian matrices A and B such that $M = A + iB$. (A matrix X is called Hermitian if $\overline{X^t} = X$).

Determinants

Problem 3.1

Let $(F_n)_n$ be the Fibonacci sequence. Using determinants, prove the identity

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n, \quad \text{for all } n \geq 1.$$

Problem 3.2

Let X and Y be $n \times n$ matrices. Prove that

$$\det(\mathcal{I}_n - XY) = \det(\mathcal{I}_n - YX).$$

Inverses

Problem 4.1

Let A and B be 2×2 matrices with integer entries such that A, $A + B$, $A + 2B$, $A + 3B$, and $A + 4B$ are all invertible matrices whose inverses have integer entries. Prove that $A + 5B$ is invertible and that its inverse has integer entries.

Problem 4.2

Let A be an $n \times n$ matrix such that there exists a positive integer k for which

$$kA^{k+1} = (k+1)A^k.$$

Prove that the matrix $A - \mathcal{I}_n$ is invertible and find its inverse.

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