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Partial Differential Equations
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(6.3) Poisson's Formula

$$\Delta_2 u = 0 \qquad (x,y) \in D_a = \{(x,y) \in \mathbb{R}^2 / x^2+y^2 < a^2\}$$
$$u(a,\theta) = h(\theta) \text{ in } \partial D_a$$

$$\Delta_2 u = u_{xx} + u_{yy} = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$$

$$r \in (0,a) \qquad \theta \in (0, 2\pi)$$

$$u(r,\theta) = R(r) T(\theta)$$

$$R'' T + \frac{1}{r} R' T + \frac{R T''}{r^2} = 0 \quad \times \frac{r^2}{RT}$$

$$\Longleftrightarrow \quad r^2 \frac{R''}{R} + \frac{r R'}{R} + \frac{T''}{T} = 0$$

$$\frac{r^2 R''}{R} + \frac{r R'}{R} = -\frac{T''}{T} = \lambda_n$$

$$\begin{cases} T'' + \lambda_n T = 0 \\ r^2 R'' + r R' - \lambda_n R = 0 \end{cases}$$

$$\Longleftrightarrow \quad T(\theta) = A\cos(\sqrt{\lambda_n}\,\theta) + B\sin(\sqrt{\lambda_n}\,\theta)$$

Since T is periodic $\Longrightarrow$ $T(\theta + 2\pi) = T(\theta)$

$$T(\theta+2\pi) = A\cos(\sqrt{\lambda_n}(\theta+2\pi)) + B\sin(\sqrt{\lambda_n}(\theta+2\pi))$$
$$= T(\theta) \quad \text{if } \sqrt{\lambda_n} = n \quad (n \in \mathbb{Z}^+)$$

$$\Longrightarrow T(\theta) = A\cos(n\theta) + B\sin(n\theta)$$

If $\lambda_n = 0 \Rightarrow \quad T(\theta) = A$ (if $n=0$)

$$r^2 R'' + nR' - n^2 R = 0$$

This in a Cauchy-Euler Equation that solves to

$$R(r) = Cr^{-n} + Dr^n$$

$$T(\theta) = A\cos(n\theta) + B\sin(n\theta)$$

$$T(\theta) = A \quad \text{if } n=0$$

$$R(r) = Cr^{-n} + Dr^{n} \quad \text{if } n \neq 0$$

If $n=0$, $r^2R'' + rR' = 0$

$$\frac{R''}{R'} = -\frac{1}{r}$$

$$\ln(R') = -\ln(r) + C$$

$$\cancel{R(r) = \frac{C}{r} + D}$$

$$R(r) = C\ln(r) + D$$

If $n \neq 0$

$$T(\theta) = A\cos(n\theta) + B\sin(n\theta)$$

$$R(r) = Cr^{-n} + Dr^n$$

$\hookrightarrow C = 0$

Because $r^{-n} \to +\infty$ as $r \to 0$

If $n=0$

$$T(\theta) = A$$

$$R(r) = C\ln(r) + D$$

$C = 0$

because u is not singular at 0.

(Harmonic means that Laplace of u is zero).

$$u(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \cancel{\emptyset}\, r^n\left(D_n A_n \cos(n\theta) + D_n B_n \sin(n\theta)\right)$$

$$u(a,\theta) = h(\theta) = \frac{A_0}{2} + \sum_{n=1}^{+\infty} a^n\left(D_n A_n \cos(n\theta) + D_n B_n \sin(n\theta)\right)$$

$$A_0 = \frac{2}{2\pi}\int_{0}^{2\pi} h(\phi)\,d\phi \qquad \cancel{D_n A_n =}$$

$$\int_0^{2\pi} h(\theta)\cos(k\theta)\,d\theta = a^k D_k A_k \underbrace{\int_0^{2\pi} \cos^2(k\theta)\,d\theta}_{\pi}$$

$$\int \cos(k\theta)\cos(n\theta) = 0 \quad (n \neq k)$$

$$\int \sin(k\theta)\cos(n\theta) = 0 \quad (n \neq k)$$

$$D_k A_k = \frac{1}{\pi a^k}\int_0^{2\pi} h(\theta)\cos(k\theta)\,d\theta$$

Similarly $D_k B_k = \dfrac{1}{\pi a_k}\displaystyle\int_0^{2\pi} h(\theta)\sin(k\theta)\,d\theta$

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)\,d\phi + \sum_{n=1}^{+\infty} r^n\left(D_n A_n \cos(n\theta) + D_n B_n \sin(n\theta)\right)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} h(\phi)\,d\phi + \sum_{n=1}^{+\infty} r^n\Bigg(\frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\cos(n\phi)\cos(n\theta)\,d\phi$$

$$+ \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\sin(n\phi)\sin(n\theta)\,d\phi\Bigg)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} h(\phi)\left[1 + 2\sum_{n=1}^{+\infty}\left(\frac{r}{a}\right)^n \underbrace{\left[\cos(n\phi)\cos(n\theta) + \sin(n\phi)\sin(n\theta)\right]}_{\cos(n(\theta-\phi))}\right]d\phi$$

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)\left(1+2\sum_{n=1}^{+\infty}\left(\frac{r}{a}\right)^n \cos(n(\theta-\phi))\right) d\phi$$

$$\cos[n(\theta-\phi)] = \frac{e^{in(\theta-\phi)} + e^{in(\theta-\phi)}}{2}$$

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)\left(1+\sum_{n=1}^{+\infty}\left(\frac{r}{a}\right)^n e^{in(\theta-\phi)} + \sum_{n=1}^{+\infty}\left(\frac{r}{a}\right)^n e^{-in(\theta-\phi)}\right)$$

$$\sum_{n=1}^{+\infty}\left[\frac{re^{i(\theta-\phi)}}{a}\right]^n = \frac{\frac{re^{i(\theta-\phi)}}{a}}{1-\frac{re^{i(\theta-\phi)}}{a}} = \frac{re^{i(\theta-\phi)}}{a-re^{i(\theta-\phi)}}$$

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)\left(1+\frac{re^{i(\theta-\phi)}}{a-re^{i(\theta-\phi)}} + \frac{re^{i(\theta-\phi)}}{a-re^{i(\theta-\phi)}}\right) d\phi$$

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)\left[\frac{a^2-r^2}{a^2-2ar\cos(\theta-\phi)+r^2}\right] d\phi$$

$$= \frac{a^2-r^2}{2\pi}\int_0^{2\pi} \frac{h(\phi)}{a^2-2ar\cos(\theta-\phi)+r^2} d\phi$$

By using polar coordinates

$$u(\vec{x}) = \frac{a^2-|\vec{x}|^2}{2\pi a}\int_{\partial D} \frac{u(\vec{x}')}{|\vec{x}-\vec{x}'|} ds \qquad (ds = a\,d\phi)$$