Problem 1

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Problem 1 (1.1): Let $X_n, n = 0, 1, 2, ...$ be a sequence of independent random variables, each of which assumed non-negative integer values. Define a sequence of partial sums:

$$S_n = \sum_{i=1}^{n} X_i \tag{1}$$

Show that $S_n, n = 0, 1, 2, ...$ is a Markov Chain.

By Definition, we have:

$$S_n = \sum_{i=1}^{n} X_i = X_n + \sum_{i=1}^{n-1} X_i \tag{2}$$

$$X_n = S_n - S_{n-1} \tag{3}$$

We know that:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{4}$$

Therefore,

$$P(S_{n+1} = s_{n+1}|S_n = s_n, S_{n-1} = s_{n-1}, S_{n-2} = s_{n-2}, ..., S_1 = s_1) =$$

$$\frac{P(S_{n+1}=s_{n+1},S_n=s_n,S_{n-1}=s_{n-1},S_{n-2}=s_{n-2},...,S_1=s_1)}{P(S_n=s_n,S_{n-1}=s_{n-1},S_{n-2}=s_{n-2},...,S_1=s_1} =$$

$$\frac{P(S_n+X_{n+1}=s_{n+1},S_{n-1}+X_n=s_n,S_{n-2}+X_{n-1}=s_{n-1},...,S_1+X_2=s_2 S_1=X_1=s_1)}{P(S_{n-1}+X_n=s_n,S_{n-2}+X_{n-1}=s_{n-1},...,S_1+X_2=s_2 S_1=X_1=s_1)} =$$

$$\frac{P(X_{n+1}=s_{n+1}-s_n,X_n=s_n-s_{n-1},X_{n-1}=s_{n-1}-s_{n-2},...,X_2=s_2-s_1,X_1=s_1)}{P(X_n=s_n-s_{n-1},X_{n-1}=s_{n-1}-s_{n-2},...,X_2=s_2-s_1,X_1=s_1)} =$$

$$\frac{P(X_{n+1}=s_{n+1}-s_n)P(X_n=s_n-s_{n-1})P(X_{n-1}=s_{n-1}-s_{n-2})...P(X_2=s_2-s_1)P(X_1=s_1)}{P(X_n=s_n-s_{n-1})P(X_{n-1}=s_{n-1}-s_{n-2})...P(X_2=s_2-s_1)P(X_1=s_1)} =$$

$$P(X_{n+1} = s_{n+1} - s_n)$$

Replacing n by $n + 1$ in Equation (3) we get:

$$X_{n+1} = S_{n+1} - S_n$$

Hence X_{n+1} is the incremental value between S_n and S_{n+1}. Therefore:

$$P(X_{n+1} = s_{n+1} - s_n) = P(S_{n+1} = s_{n+1} | S_n = s_n)$$

Therefore, we get:

$$P(S_{n+1} = s_{n+1} | S_n = s_n) = P(S_{n+1} = s_{n+1} | S_n = s_n, S_{n-1} = s_{n-1}, S_{n-2} = s_{n-2}, ..., S_1 = s_1)$$

Hence, S_n is a Markov Chain.