Markov Processes: Problem Set

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The problems are taken from ”Elemeets of the Theory of Markov Processes with Applications” by A. T. Bharucha-Reid.

Problem 1 (1.1): Let $X_n, n = 0, 1, 2, ...$ be a sequence of independent random variables, each of which assumed non-negative integer values. Define a sequence of partial sums:

$$S_n = \sum_{i=1}^{n} X_i \tag{1}$$

Show that $S_n, n = 0, 1, 2, ...$ is a Markov Chain.

Problem 2 (1.2): Prove the the following propositions: Two states i and j of a Markov Chain *communicate* if and only if $L_{ij} > 0$ and $L_{ji} > 0$.

Problem 3 (1.3): Classify the states of the Markov chains whose transition probabilities are given by:
(a) $p_{02} = 1; p_{11} = 1; p_{i,i-1} = p_{i,i+1} = \frac{1}{2}$
(b) $p_{00} = \frac{1}{2}; p_{01} = \frac{1}{2}; p_{i,i-1} = p_{i,i+1} = \frac{1}{2}$
(c) $p_{00} = \frac{1}{3}; p_{01} = \frac{2}{3}; p_{i,i-1} = \frac{1}{3}; p_{i,i+1} = \frac{2}{3}$

Problem 4 (1.4): Calculate the higher moments and cumulants of X_n, where X_n is a simple branching process.

Problem 5 (2.2): Let $X(t), t \geq 0$ be a time-homogeneous stochastic process with independent increments and let:

$$Y_i(t) = P([X(t) - X(0)] = i)$$

satisfy the conditions:

$$\lim_{t \to 0} \frac{Y_1(t)}{t} = \lambda > 0; \lim_{t \to 0} \frac{1 - Y_0(t) - Y_1(t)}{t} = 0 \tag{2}$$

Show that:

$$Y_i(t) = \frac{(\lambda t)^i}{i!} e^{-\lambda t} \tag{3}$$

for i=0, 1, ...

Problem 6 (2.3): If F(s,t) is the generating function associated with the random variable X(t), show that F(1/s,t) is the generating function associated with -X(t).

Problem 7 (2.5): Solve the equations for the birth process with:

$$\lambda_x = \lambda + yx,\ \ x = 0, 1, ... \tag{4}$$

and the initial conditions $P_1(0) = 1$, $P_x(0) = 0$, for $x \neq 1$.

Problem 8 (1.8): A Markov Chain with a Transition Matrix P is said to be periodic with period ω if $P^{a+\omega} = P^a$ and ω is the smallest positive integer with this property. Determine the limit matrix Π for periodic chains.

Problem 9 (1.10): Consider a random walk process in which a moving particle can occupy any of the points $i = a + 1, a + 2, ..., b - 1$ on the segment [a, b]. If the particle is at position i at time t, then it with probability p_i it will be at $i+1$ at time $t+1$ and with probability q_i it will be at $i-1$ at time $t+1$, where $p_i + q_i = 1$ for $i = a + 1, a + 2, ..., b - 1$. Determine the probability, say $f(a; x)$ $x \in (a, b)$ that a particle starting at position x at $t = 0$ will land at the position a before landing at position b. Consider x and b to be fixed and a variable.

Problem 10 (1.6): Prove the fundamental theorem concerning branching processes by utilizing the theory of absorption probabilities.

Problem 11 (2.10): Consider a birth-and-death process with parameters λ_x and μ_x. Let T_n denote the time required for the random variable to increase from n to $n + 1$ and let $T_n^* = E\{T_n\}$. T_n^* is the conditional expected time, conditioned upon non-absorption or extinction. Show that $T_n^* = \frac{1}{\lambda_n} + \frac{\mu_n}{\lambda_n} T_{n-1}^*$

Problem 12 (2.13): The Coefficient of Variation of the random variable $X(t)$ is defined as:

$$\nu[X(t)] = \frac{D[X(t)]}{E[X(t)]}$$

the ration of standard deviation and mean of $X(t)$. Determine the Coefficient of Variation and its Asymptotic Behavior for (a) Poisson Process, (b) Simple Birth Process and (c) Birth-and-Death Process.