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PDE

Chapter 3 : Reflection and Sources

3.1 : The Heat Equation

(1) $\begin{cases} u_t - k u_{xx} = 0 & x \in (0, \infty) \\ u(x,0) = \phi(x) & t \in (0, \infty) \\ u(0,t) = 0 \end{cases}$

Theorem (*): There exists a unique solution of (1):

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_0^\infty \left[e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}} \right] \phi(y)\, dy$$

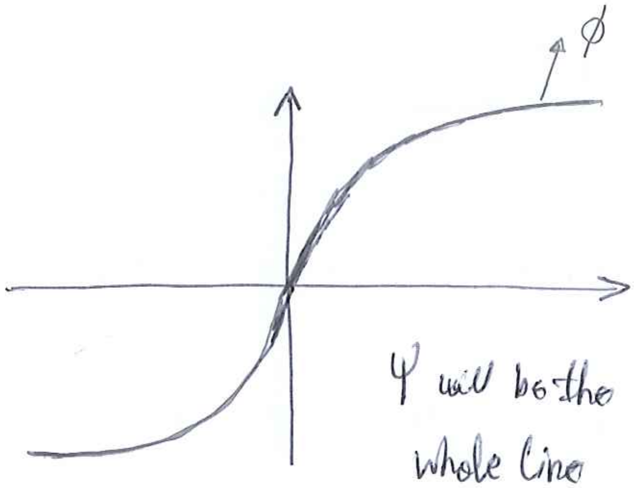
Proof:

$$\begin{cases} V_t - k V_{xx} = 0 & , x \in R \\ V(x,0) = \psi(x) \end{cases}$$

$$V(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\,\psi(y)\,dy$$

$$S(x,t) = \frac{1}{\sqrt{4k\pi t}} e^{-\frac{x^2}{4kt}}$$

ψ is the odd extension of ϕ.



ψ will be the whole line

$$\phi_{odd}(x) = \begin{cases} \phi(x) & x \geq 0 \\ -\phi(-x) & x < 0 \end{cases}$$

If $x \geq 0$, $v(x,t) = u(x,t)$

$$u(x-y,t) = \int_{-\infty}^{0} S(x-y,t)\,\phi_{odd}(y)\,dy + \int_{0}^{+\infty} S(x-y,t)\,\phi_{odd}(y)\,dy$$

$$= \int_{-\infty}^{0} S(x-y,t)(-\phi(-y))\,dy + \int_{0}^{+\infty} S(x-y,t)\,\phi(y)\,dy$$

$\cancel{u(x,t) = -\int}$

$$\left.\begin{aligned} z &= -y \\ dz &= -dy \end{aligned}\right] \to \text{Transformation}$$

$$u(x,t) = -\int_{-\infty}^{0} S(x+z,t)\,\phi(z)\,dz$$

$$+ \int_{0}^{+\infty} S(x-y,t)\,\phi(y)\,dy$$

This leads to Theorem($\star$)

Proposition :

If $x \geq 0$ $\quad V(x,t) = u(x,t)$

If $x \leq 0$ $\quad V(x,t) = -u(-x,t)$

Obvious case for $x \geq 0$

Now $x \leq 0$,

$$V(x,t) = \int_{-\infty}^{+\infty} S(x-y,t)\,\phi_{odd}(y)\,dy$$

Suppose that $z \leq 0$ and $z = -x$

$$V(z,t) = \int_{0}^{\infty} S(z-y,t)\,\phi_{odd}(y)\,dy - \int_{0}^{+\infty} S(z+y,t)\,\phi(y)\,dy$$

[Same operations as before]

Setting $z = -x$

$$V(x,t) = \int_{0}^{\infty} S(-x-y,t)\,\phi_{odd}(y)\,dy - \int_{0}^{+\infty} S(-x+y,t)\,\phi_{odd}(y)\,dy$$

$$= -u(x,t)$$

$$= -u(-z,t)$$

$\therefore$ V is the odd extension of u.

Neriman Boundary Condition (NBC)

$$u_x(0,t) = 0$$

Proof:

$\Rightarrow$ ~~φ will be an even function~~

$$\begin{cases} v_t - k v_{xx} = 0 \\ v(x,0) = \varphi(x) \end{cases}$$

$\Rightarrow$ φ will be an even function

3.1 The Wave Equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & , x \in (0, +\infty) \\ u(x,0) = \phi(x), \ u_t(x,0) = \psi(t) \\ u(0,t) = 0 \end{cases}$$

Theorem:

$$u(x,t) = \begin{cases} \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds, & x > c|t| \\ \frac{1}{2}\left[\phi(x+ct) - \phi(x-ct)\right] + \frac{1}{2c}\int_{ct-x}^{ct+x} \psi(s)\,ds, & 0 \le x \le c|t| \end{cases}$$