Chapter 3: Reflections and Sources (15/March)

3.2 Reflection of Waves

$$\begin{cases} u_t - k u_{xx} = 0 \\ u(x,0) = \phi(x) \end{cases} \qquad u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\,\phi(y)\,dy$$

$$u_t - k u_{xx} = 0 \qquad x \in [0,+\infty)$$

$$u(x,0) = \phi(x)$$

$$u(0,t) = 0 \quad (D.B.C)$$

For $x \in \mathbb{R}$

$$\begin{cases} v_t - k v_{xx} = 0 \\ v(x,0) = \varphi(x) \end{cases}$$

Hence $\varphi = \begin{cases} \phi(x) & , x \geq 0 \\ -\phi(-x) & , x < 0 \end{cases}$

For the wave equation:

$$v_{tt} - c^2 v_{xx} = 0 \qquad x \in [0,+\infty]$$

$$v(x,0) = \phi$$

$$v_t(x,0) = \psi$$

Adding DBT: $v(0,t) = 0$

Theorem:

The unique solution:

$$V(x,t) = \begin{cases} \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy \right. , & x > c|t| \\ \\ \frac{1}{2}\left[\phi(x+ct) - \phi(x-ct) + \frac{1}{2}\int_{ct-x}^{ct+x} \psi(y)\,dy \right. , & 0 < x < c|t| \end{cases}$$

$$u_{tt} - c^2 u_{xx} = 0$$

$$u(x,0) = \phi(x)$$

$$u_t(x,0) = \psi(x)$$

$$u(x,t) = \frac{1}{2}\left[\phi_{odd}(x+ct) + \phi_{odd}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{odd}(y)\,dy$$

$$\phi_{odd}(y) = \begin{cases} \phi(y), & y \geq 0 \\ -\phi(-y), & y < 0 \end{cases} \qquad \psi_{odd}(y) = \begin{cases} \psi(y), & y \geq 0 \\ -\psi(-y), & y < 0 \end{cases}$$

Now we consider on a finite interval,

$$V_{tt} - c^2 V_{xx} = 0 \qquad x \in [0, L]$$

$$V(x,0) = \phi$$

$$V_t(x,0) = \psi$$

$$V(0,t) = 0, \quad V(L,t) = 0$$

$$\psi, \phi : [0, L] \to \mathbb{R}$$

Understand this