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A. Stochastic Processes

A (Discrete Time) stochastic process is a simple sequence $X_0, X_1, X_2, \ldots$ of random variables defined on some fixed probability triple $(\Omega, \mathcal{F}, P)$. The random variables $\{X_n\}$ are typically not independent. In this context we often think of n as representing time, and X_n represents the value of a random quantity at the time n.

Theorem (A1): Let U be a random variable whose distribution is Lebesgue measure on $[0,1]$. Let F be any cumulative distribution function, and set $\phi(u) = \inf\{x ; F(x) \geq u\}$ for $0 < u < 1$. Then $P(\phi(U) \leq x) = F(x)$ for each $x \in R$; in words, the cumulative distribution function of $\phi(U)$ is F.

Proof: Since F is right continuous (cumulative) we can say that $\inf\{x ; F(x) \geq u\} = \min\{x ; F(x) \geq u\}$. It will follow that $\phi(u) \leq x \Leftrightarrow u \leq F(x)$. Hence, since $0 \leq F(x) \leq 1$, we obtain that $P(\phi(U) \leq x) = P(U \leq F(x)) = F(x)$. $\square$

Theorem (A2): Let $\mu_1, \mu_2, \mu_3, \ldots$ be any sequence of Borel probability measures on R. Then there exists a probability space $(\Omega, \mathcal{F}, P)$ and random variables $X_1, X_2, X_3 \ldots$ defined on $(\Omega, \mathcal{F}, P)$ such $\{X_n\}$ are independent and $\mathcal{L}(X_n) = \mu_n$.

Proof: Let $(\Omega, \mathcal{F}, P)$ be infinite independent fair coin tossing so that $r_1, r_2, r_3 \dots$ are i.i.d. with $P(r_i = 0) = P(r_i = 1) = \frac{1}{2}$. Let $\{Z_{ij}\}$ be a two-dimensional array filled by these r_i as follows:

$$\begin{pmatrix} Z_{11} & Z_{12} & Z_{13} & \cdots \\ Z_{21} & Z_{22} & Z_{23} & \cdots \\ Z_{31} & Z_{32} & Z_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \equiv \begin{pmatrix} r_1 & r_3 & r_6 & \cdots \\ r_2 & r_5 & \cdots & \\ r_4 & \cdots & \cdots & \\ \vdots & \vdots & & \end{pmatrix}$$

Hence $\{Z_{ij}\}$ are independent with $P(Z_{ij} = 0) = P(Z_{ij} = 1) = \frac{1}{2}$.

Then for each $n \in \mathbb{N}$ we set $U_n = \sum_{k=1}^{\infty} \frac{Z_{nk}}{2^k}$. Since Z_{ij} are independent, U_n is also independent. Also by the way the U_n were constructed we have

$$P\left(\frac{j}{2^k} \le U_n < \frac{j+1}{2^k}\right) = \frac{1}{2^k}, \quad k \in \mathbb{N} \text{ and } 0 \le j < 2^k.$$

By the additivity and continuity of probabilities, this implies that $P(a \le U_n < b) = b - a$, whenever $0 \le a < b \le 1$. Therefore, each U_n follows a uniform distribution (Lebesgue Measure) on $[0, 1]$.

Finally, we set $F_n(x) = \mu_n((-\infty, x])$ for $x \in \mathbb{R}$ and set $\phi_n(u) = \inf\{x ; u \le F_n(x)\}$ for $0 < u < 1$ and set $X_n = \phi_n(U_n)$. Then $\{X_n\}$ are independent by definition since U_n are independent. And by Theorem (1) we get $X_n \sim \mu_n$, as required.

Theorem (A3): (The Bounded Convergence Theorem) Let $\{X_n\}$ be a sequence of random variables, with $\lim X_n = X$. Suppose there is $K \in R$ such that $|X_n| \leq K$ for all $n \in \mathbb{N}$. Then $E(X) = \lim E(X_n)$.

Proof: We have from triangle inequality that,

$$|E(X) - E(X_n)| = |E(X - X_n)| \leq E(|X - X_n|).$$

We will show that $E(|X - X_n|) \to 0$ as $n \to \infty$. Thus for $\epsilon > 0$ we set,

$A_n = \{ w \in \Omega ; |X(w) - X_n(w)| > \epsilon \}$. Then we have,

$|X(w) - X_n(w)| \leq \epsilon + 2K 1_{A_n}(w)$, so that

$$E(|X - X_n|) \leq \epsilon + 2KP(A_n).$$

However we have the following,

$$\begin{aligned} \limsup E(|X - X_n|) &\leq \epsilon + 2K \limsup P(A_n) \\ &\leq \epsilon + 2KP(\limsup A_n) \\ &= \epsilon, \end{aligned}$$

Since $|X(w) - X_n(w)| \to 0$ for all $w \in \Omega$ as $n \to \infty$.

Therefore $\limsup_{n \to \infty} A_n$ is the empty set. Hence we get

$E(|X - X_n|) \to 0$ as claimed. $\square$

B. Markov Chains

A Markov Chain consists of three ingredients: a state space S which is a finite or a countable set, an initial distribution $\{v_i\}_{i\in S}$ and transition probabilities $\{p_{ij}\}_{i,j\in S}$, $\sum_j p_{ij} = 1$. We can say that

$$P(X_1=j) = \sum_{i\in S} P(X_0=i, X_1=j) = \sum_{i\in S} v_i p_{ij}.$$

Theorem (B1): (Markov Chain Existence Theorem) Given a non-empty countable set S, and non-negative numbers $\{v_i\}_{i\in S}$ and $\{p_{ij}\}_{i,j\in S}$, with $\sum_j v_j = 1$ and $\sum_j p_{ij} = 1$ for each $i\in S$, there exists a probability triple $(\Omega, \mathcal{F}, P)$ and random variables $X_0, X_1, \ldots$ defined on $(\Omega, \mathcal{F}, P)$ such that

$$P(X_0=i_0, X_1=i_1, \ldots, X_n=i_m) = v_{i_0} p_{i_0 i_1} \cdots p_{i_{n-1} i_n}.$$

for all $n\in N$ and all $i_0 \ldots i_n \in S$.

Proof: We let $(\Omega, \mathcal{F}, P)$ be a Lebesgue measure on $[0,1]$. We construct the random variables $\{X_n\}$ as follows:

1. Partition $[0,1]$ into intervals $\{I_i^{(0)}\}_{i\in S}$ with $\text{length}(I_i^{(0)}) = v_i$
2. Partition each $I_i^{(0)}$ into intervals $\{I_{ij}^{(0)}\}_{i,j\in S}$ with length $(I_{ij}^{(1)}) = v_i p_{ij}$.
3. Inductively partition $[0,1]$ into intervals $\{I^{(n)}_{i_0 i_1 \ldots i_n}\}_{i_0 \ldots i_n \in S}$ such that $I^{(n)}_{i_0 \ldots i_n} \subseteq I^{(n-1)}_{i_0 \ldots i_{n-1}}$ for all $n\in N$
4. Define X_n saying that $X_n(\omega) = i_n$ if $\omega \in I^{(n)}_{i_0 i_1 \ldots i_n}$ for some choice of $i_0 \ldots i_{n-1} \in S \Longrightarrow \{X_n\}$ have desired properties $\square$