Partial Differential Equations

2.1 Done

2.2 (a) Causality

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

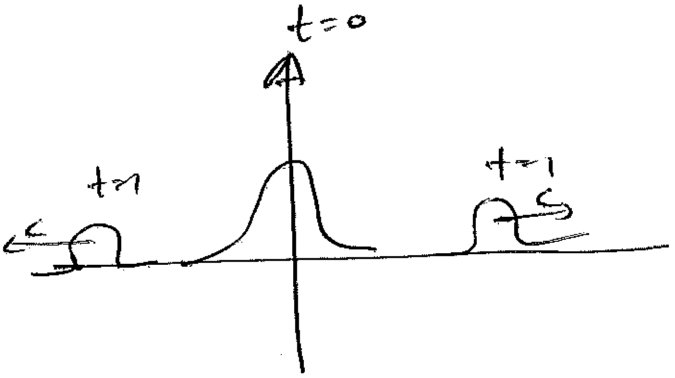
Soln:

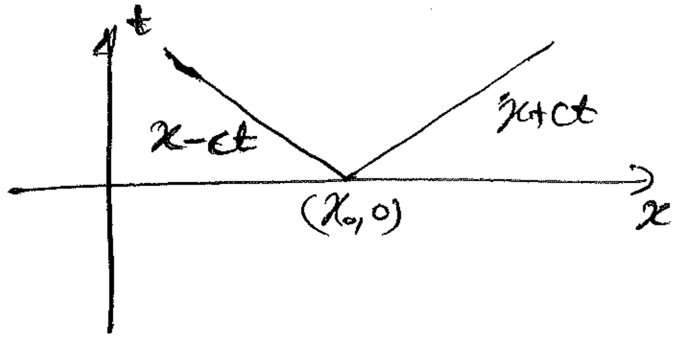
$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds$$

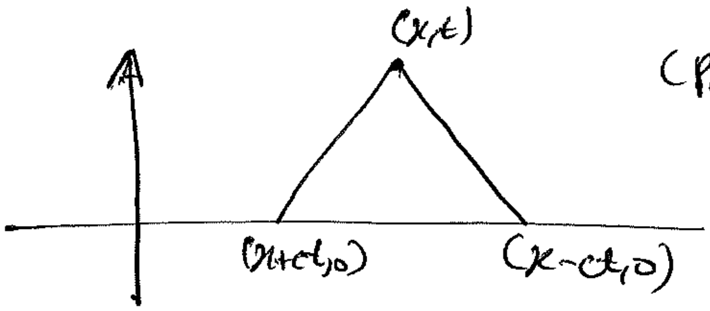
$\hookrightarrow$ d'Alembert Formula.

Suppose $\phi = \psi = 0, \quad |x| > R$

$$u(x,t) = 0 \quad \text{if} \quad |x| > R + ct$$







(Past history of the point).

2.2 (b) Energy

$$KE = \frac{1}{2} m v^2 = \frac{1}{2} \rho \int_{-\infty}^{\infty} u_t^2 \, dx$$

$\phi = \varphi = 0 \quad , |x| > R$

$\Rightarrow u = 0 \quad |x| > R + ct$

$$\frac{dKE}{dt} = \frac{1}{2} \rho \int_{-\infty}^{\infty} 2 u_t u_{tt} \, dx = \frac{1}{2} \rho \int_{-\infty}^{\infty} \frac{d}{dt} \left[(u_t)^2 \right] dx$$

$$= \rho \int_{-\infty}^{\infty} u_t \cdot u_{tt} \, dx$$

Since $\rho u_{tt} = T u_{xx} \Rightarrow \frac{dKE}{dt} = T \int_{-\infty} u_{xx} \cdot u_{t} \, dx$

$$\boxed{\begin{aligned} KE &= \frac{1}{2} \rho \int_{-\infty}^{\infty} u_t^2 \, dx \\ PE &= \frac{1}{2} \int_{-\infty}^{\infty} T u_x^2 \, dx \end{aligned}}$$

$$= -T \int_{-\infty}^{\infty} u_x u_{tx} \, dx + \underbrace{\left[u_x u_t \right]_{-\infty}^{+\infty}}_{=0}$$

$$= -\frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} T u_x^2 \, dx$$

Since, $\frac{1}{2} \frac{d}{dt} (u_x^2) = u_x u_{xt}$

Hence $\frac{d}{dt} \Big[\underbrace{KE + PE}_{E} \Big] = 0$

We also have,

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} (\rho u_t^2 + T u_x^2)\, dx = \frac{1}{2}\int_{-\infty}^{\infty} \left(\rho \psi^2 + T(\phi')^2\right) dx$$

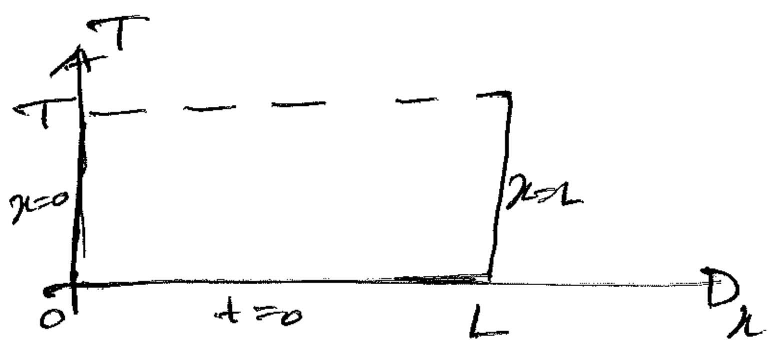
$$= E(0) \qquad = \frac{1}{2}\int_{-R}^{R} \left(\rho\psi^2 + T(\phi')^2\right) dx$$

2.3: The Diffusion Equation

$$(1) \quad \begin{cases} u_t - k u_{xx} = 0 & x \in (0, L] \text{ and } t \in [0, +T] \\ u(x,0) = \phi(x), \; u(0,t) = g(t) & \\ \text{Dirichlet B.C.} & (\text{See } 1.4) \\ u(L,t) = \psi(t) & \end{cases}$$

○ Theorem (Maximum Principle):

If u is a ~~sol~~ sol of (1), then the maximum of u is reached only at $t=0$ or on the lateral side ($x=0$ or $x=L$).



$R = [0, L] \times [0, T]$

Proof:

Suppose $(x_0, t_0) \in R$ and u reaches its max at (x_0, t_0)

$$u_t(x_0,t_0) = u_x(x_0,t_0) = 0 \quad \text{---} \ (2)$$

Hence (x_0,t_0) is a critical point.

$$u_{xx}(x_0,t_0) \le 0 \quad (\text{Max point}).$$

Suppose $u_{xx}(x_0,t_0) \neq 0 \Rightarrow u_{xx}(x_0,t_0) < 0$

This is a contradiction because

$$u_t(x_0,t_0) = k\,u_{xx}(x_0,t_0) < 0$$

$$(u_t(x_0,t_0) = 0) \text{ by } (2).$$

$\therefore$ u reaches its max on boundary

$M :=$ max of u on $\{t=0\}$, $\{x = 0, L\}$

Let $\varepsilon > 0$ $\quad v(x,t) = u(x,t) + x^2 \varepsilon$