The Diffusion Equation

Theorem: Consider u the solution of:

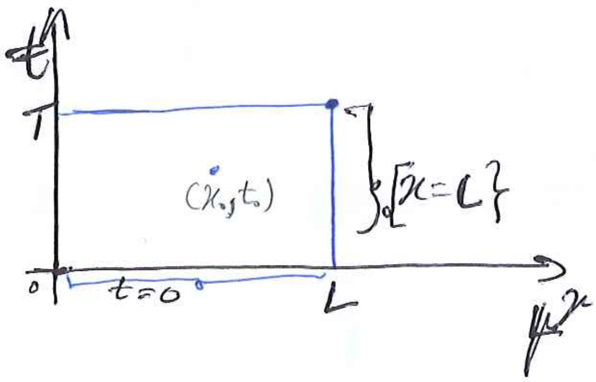
$$u_t = k u_{xx} \qquad x \in [0,L] , t [0,T]$$

$$u(x,0) = \phi(x)$$

Dirichlet B.C.

Then u attains its maximum at either $\{t=0\}$ or $\{x=0\}$ or $\{x=L\}$

Proof:



Suppose that the maximum is not on boundary (x_0,t_0)

$$\Rightarrow \quad d_t u(x_0,t_0) = 0 \quad (1) \quad \Longrightarrow \quad u_{xx}(x_0,t_0) \le 0$$

$$d_x u(x_0,t_0) = 0 \quad (2)$$

(Maximum)

Suppose $u_{xx}(x_0,t_0) \neq 0$

$$u_{xx}(x_0,t_0) < 0 \quad \Longrightarrow \quad u_t = k u_{xx} < 0$$

Contradiction with (2).

$$V(x,t) = u(x,t) + \varepsilon x^2 , \quad \varepsilon > 0$$

$$\Longrightarrow \quad V_{xx} = u_{xx} + 2\varepsilon \ ; \quad V_t = u_t$$

Then $V_t - kV_{tt} = U_t - k(U_{xx} + 2\varepsilon) = \underbrace{U_t - kU_{xx}}_{=0} - 2k\varepsilon$

$\cancel{V_t - kV_{xx} =}$

$$\underbrace{V_t - kV_{xx} = -2k\varepsilon < 0}$$

Inequality
Diffusion

M is the maximum of u in $\underset{Y_1}{[t=0]}, \underset{Y_2}{[x=0]}, \underset{Y_3}{[x=L]}$.

We will prove that $u(x,t) \leq M$, for all $(x,t) \in R$

☆ This is equivalent to prove that $\underline{V(x,t) \leq M + \varepsilon L^2}$, $\forall (x,t) \in R$

Because, $u(x,t) + \varepsilon x^2 \leq M + \varepsilon L^2$

$u(x,t) \leq M + \varepsilon(L^2 - x^2)$

As this is true for all $\varepsilon \Longrightarrow u(x,t) \leq M$.

Suppose that V attains max in interior of $R; (x_0, t_0)$

$V_t(x_0, t_0) = 0$
$V_x(x_0, t_0) = 0$ and $V_{xx}(x_0, t_0) \leq 0$

Now if $V_t = 0$ it contradicts the Inequality Diffusion.

$\therefore$ Hence V does not take maximum in interior.

Let $(x,t) \in R_T = \{(x,t) \in R \quad 0 \leq x \leq L, t = T\}$

R_T is top edge

Theorem: If u attains its maximum at (x_0, y_0) in $\mathring{\Omega} = \Omega \setminus \partial\Omega$

Then (x_0, y_0) is a critical point.

i.e. $\vec{\nabla} u(x_0, y_0) = \begin{pmatrix} d_x u(x_0, y_0) \\ d_y u(x_0, y_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Suppose that v attains its maximum at the top edges of (x_0, T).

$$\frac{dv}{dt} = \frac{v(x_0, T) - v(x_0, T-\delta)}{\delta} > 0 \quad \text{—} (4)$$

It should be positive because (x_0, T) is maximum and values before it should ~~be~~ have a positive derivative.

$$v_x(x_0, T) = 0 \quad , \quad v_{xx}(x_0, T) \leq 0 \quad \text{—} (3)$$

Think $v(x, T) = f(x) \quad 0 \leq x \leq l$

max at $f(x_0)$

Hence $f'(x_0) = 0$

$f''(x_0) \leq 0$

~~From Inequality Diffusion, and (3)~~

From (3) and (4) we get $v_t - k v_{xx} > 0$

This contradicts the Inequality Diffusion.

$\therefore$ This prove the Diffusion Equation claim we made.

Theorem: $\begin{cases} u_t - k u_{xx} = 0 \\ u(x,0) = \phi(x) \end{cases}$

has a unique solution.

Proof-

Supp u_1 and u_2 are two solutions

$$W = u_1 - u_2$$

$$W_t - k W_{xx} = 0$$

$$W(x,0) = 0 \quad (\text{as } W = u_1 - u_2)$$

$$\Longrightarrow W(x,0) \leq 0$$

Tho $(-W)_t - k(-W)_{xx} = 0$

$$(-W)(x,0) = 0 \Longrightarrow (-W) \leq 0$$

$$W \leq 0, (-W) \leq 0 \Longrightarrow \boxed{W = 0}$$

uniqueness