The Wave Equation

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1 The Wave Equation

As mentioned earlier in the notes, the wave equation is popularly expressed as:

$$u_{tt} = c^2 u_{xx} \quad for \ -\infty < x < +\infty \tag{1}$$

Physically, it can be imagined as a very long string. The Wave Equation can be factored as follows:

$$u_{tt} = c^2 u_{xx} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = 0 \tag{2}$$

Considering Equation (2), the general solution is given by:

$$u(x,t) = f(x+ct) + g(x-ct) \tag{3}$$

where f and g are two arbitrary (twice differentiable) functions of a single variable.

2 Initial Value Problem

The I.V.P. is to solve the Wave Equation in (1) with the initial conditions:

$$u(x,0) = \phi(x) \qquad u_t(x) = \psi(x) \tag{4}$$

where ϕ and ψ are arbitrary functions of x. The data in (4) will be used to derive the solution of (3). By setting $t = 0$ in (3) we get:

$$\phi(x) = f(x) + g(x) \tag{5}$$

And by differentiating (3) we get:

$$\psi(x) = cf'(x) - cg'(x) \tag{6}$$

We introduce a change of variable from x to s in (5) and (6). We differentiate (5) and divide (6) by c to get $\phi' = f' + g'$ and $\frac{1}{c}\psi = f' - g'$. Manipulating the last two equations and integrating the results gives:

$$f' = \frac{1}{2}\left(\phi' + \frac{\psi}{c}\right) \Rightarrow f(s) = \frac{1}{2}\phi(s) + \frac{1}{2c}\int_0^s \psi + A \tag{7}$$

and

$$g' = \frac{1}{2}\left(\phi' - \frac{\psi}{c}\right) \Rightarrow g(s) = \frac{1}{2}\phi(s) - \frac{1}{2c}\int_0^s \psi + B \tag{8}$$

where in equations (7) and (8), A and B are constants of integration. Now because of (5) we get $A + B = 0$. Substituting $s = x + ct$ for f and $s = x - ct$ for g yields the solution formula for the initial value problem:

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\ ds \tag{9}$$

3 Example

Solve $u_{tt} = c^2 u_{xx}$, $u(x,0) = e^x$, $u_t(x,0) = \sin x$

We see that in this case

$$\phi(x) = e^x \text{ and } \psi(x) = \sin x$$

The we get:

$$\int_{x-ct}^{x+ct} \sin s\ ds = \cos(x+ct) - \cos(x-ct) \tag{10}$$

Using the integration in (10) we get the solution as:

$$u(x,t) = \frac{1}{2}(e^{x+ct} + e^{x-ct}) + \frac{1}{2c}(\cos(x+ct) - \cos(x-ct)) \tag{11}$$