# The Wave Equation II

Osama K Mahmood
okm@osamakmahmood.com

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## 1 Causality

The effect of an initial velocity $\psi$ will be that the wave will spread out at a speed $\leq c$. This means that it can only effect the wave in the shaded region of Figure 1. This is known as the *Principle of Causality*.

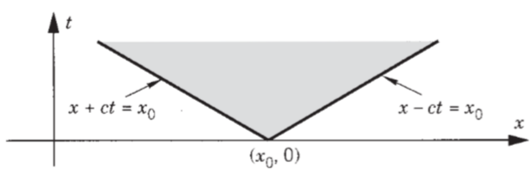


**Figure 1**

In other words, the value of $u$ at the point $(x, t)$ is affected by the value of $\phi$ at $x \pm t$ and the value of $\psi$ in the interval $[x - ct, x + ct]$. We call the interval $(x - ct, x + ct)$ as the *interval of dependence* on $t = 0$. The shaded area in triangle $\Delta$ in Figure 2 is referred to as the *Domain of Dependence* or the *Past History* of the point $(x, t)$.

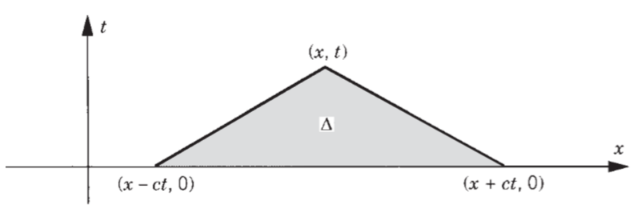


**Figure 2**

The Domain of Dependence is bounded by the characteristic lines that pass through $(x, t)$.

## 2 Energy

Imagine an infinite string with constants $\rho$ and $T$. Then $\rho u_{tt} = T u_{xx}$ for $-\infty < x < +\infty$. We know that the Kinetic Energy is given by:

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}\ \rho \int_{-\infty}^{+\infty} u_t^2\ dx \tag{1}$$

The Potential Energy is given by the following formula:

$$PE = \frac{1}{2}\int_{-\infty}^{+\infty} T u_x^2\ dx \tag{2}$$

Combining (2) with (3) we get that: We assume that $\phi = \psi = 0$ for $|x| > R$ and therefore according to the *Principle of Causality*, $u$ will be also zero on $|x| > R + ct$. Differentiating (1) before some other substitutions gives:

$$\begin{aligned}
\frac{dKE}{dt} &= \frac{1}{2}\rho \int_{-\infty}^{+\infty} \frac{d}{dt}(u_t^2)dx \\
&= \rho \int_{-\infty}^{+\infty} u_t u_{tt} dx \\
&= T \int_{-\infty}^{+\infty} u_{xx} u_t dx \quad (\rho u_{tt} = T u_{xx}) \\
&= -T \int_{-\infty}^{+\infty} u_{tx} u_x + [u_x u_t]_{-\infty}^{+\infty} dx \\
&= -\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{+\infty} T u_x^2 dx \quad (\frac{1}{2}\frac{d}{dt}u_x^2 = u_{tx}u_x)
\end{aligned} \tag{3}$$

Combining (1) with (2) gives:

$$E = \frac{1}{2}\int_{-\infty}^{+\infty} (\rho u_t^2 + T u_x^2)\ dx \tag{4}$$

Getting the derivative of equation (4) yields:

$$\frac{d}{dt}[KE + PE] = 0 \tag{5}$$

As we can note that $\frac{d}{dt}KE = -\frac{d}{dt}PE$. Hence Energy is a constant independent of $t$. This is the *Law of Conservation of Energy*.

## 3 Example

Show that the wave equation has the following invariance properties.

(a) Any translate $u(x-y,t)$, where $y$ is fixed, is also a solution.

(b) Any derivative, say $u_x$, of a solution is also a solution.

(c) The dilated function $u(ax,at)$ is also a solution, for any constant $a$.

identity

**Solution**

(a) $(u(x-y,t))_{tt} = u_{tt}(x-y,t) = c^2 u_{xx}(x-y,t) = c^2\,(u(x-y,t))_{xx}$.

(b) $(u_x(x,t))_{tt} = u_{xtt}(x,t) = c^2 u_{xxx}(x,t) = c^2\,(u_x(x,t))_{xx}$.

(c) $(u(ax,at))_{tt} = a^2 u_{tt}(ax,at) = a^2 c^2 u_{xx}(ax,at) = c^2\,(u(ax,at))_{xx}$. □