Chap 3 | 31/March

3.4 Waves and Sources

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u(x,0) = \phi \\ u_t(x,0) = \psi \end{cases} \qquad (1)$$

Theorem:

The unique solution of (1) is

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

$$+ \frac{1}{2c}\iint_{\Delta} f \, ds$$

Proof:

The solution of

$$\alpha u_{tt} + \beta u_{tx} + \gamma u_{xx} = 0$$

$$u(x,t) = F(\quad) + G(\quad)$$

We will have to solve the equation:

$$\alpha r^2 + \beta r + \gamma = 0$$

$$\Delta = \beta^2 - 4\alpha\gamma$$

$$A = \frac{-\beta + \sqrt{\Delta}}{2\alpha}$$

$$B = \frac{-\beta - \sqrt{\Delta}}{2\alpha}$$

$$\Rightarrow (\partial_t - A\partial_x)(\partial_t - B\partial_x) = 0$$

$$\left.\begin{aligned} \xi &= t - Ax \\ \eta &= t - Bx \end{aligned}\right\} \text{Change of variables.}$$

~~Proof of (1)~~

Chapter 4

Boundary Conditions

4.1 Separation of Variables and Dirichlet Boundary Condition

$$u_{tt} - c^2 u_{xx} = 0 \qquad u \in [0, L] \quad t \in \mathbb{R}$$

$$u(x,0) = \phi$$

$$u_t(x,0) = \psi$$

def.

V is a separated fct if

$$V(x,t) = X(x)T(t)$$

$$u_{tt} = c^2 u_{xx}$$

$$X(x)T''(t) = c^2 X'' T$$

$$\frac{T''}{c^2 T} = \frac{X''}{X} = \lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ T'' + \lambda c T = 0 \end{cases}$$

Claim: $\lambda > 0$

$$\lambda = \beta^2$$

$$X'' + \beta^2 X = 0$$

$$X(x) = C\cos(\beta x) + D\sin(\beta x)$$

$$T'' + \beta^2 c^2 T = 0$$

$$T(t) = A\cos(c\beta t) + B\sin(c\beta t)$$

$$\begin{cases} X(0) = 0 \\ X(L) = 0 \end{cases} \iff \begin{array}{l} C = 0 \\ D\sin(\beta L) = 0 \end{array}$$

Put $\beta_n L = \pi n \Rightarrow \beta_n = \frac{\pi n}{L}$

$$U_n(x,t) = D_n \sin\left(\frac{\pi n x}{L}\right)\left[A_n\cos\left(\frac{c\pi n t}{L}\right) + B_n\sin\left(\frac{c\pi n t}{L}\right)\right]$$

$$U_N = \sum_{n=0}^{N} U_n$$

$$\phi(x) = \sum_{n=0}^{N} \gamma_n \sin\left(\frac{\pi n x}{L}\right)$$

$$U_t(x,0) = \psi(x) = \sum_{n=0}^{N} B_n\left(\frac{n\pi c}{L}\right)\sin\left(\frac{\pi n x}{L}\right)$$

$$\phi(x) = \sum_{n=0}^{+\infty} \phi^{(n)}(x_0) \frac{(x-x_0)^n}{n!}$$

$$\phi(x) = \sum_{n=0}^{+\infty} A_n \cos(\beta_n x)$$ (We do not need regularity)

$$L^2([0,L]) = \left\{ f : \int_0^L |f|^2 < +\infty \right\}$$

$$\phi(x) = \sum_{n=0}^{\infty} \gamma_n \sin\left(\frac{\pi n x}{L}\right)$$

$\cancel{\Rightarrow u(x,t) = \sum_{n=0}^{\infty} D_n}$

$$\psi(x) = \sum_{n=0}^{\infty} \alpha_n \left(\frac{n\pi c}{L}\right) \sin\left(\frac{\pi n c}{L}\right)$$

$$\Rightarrow u(x,t) = \sum_{n=0}^{\infty} D_n \sin\left(\frac{\pi n x}{L}\right)\left[A_n \cos\left(\frac{c\pi n t}{L}\right) + B_n \sin\left(\frac{c\pi n t}{2}\right)\right]$$

Solving the Heat Equation in the same way

$$\begin{cases} u_t - K u_{xx} = 0 & x \in [0,L] \quad t \in \mathbb{R}_+ \\ u(x,0) = \phi \\ u(0,t) = u(L,t) = 0 \end{cases}$$

Expressing $u(x,t) = X(x)T(t)$

We get

$$\frac{T'}{kT} = \frac{X''}{X} = \lambda$$

$$X'' + \lambda X = 0$$

$$T' + k\lambda T = 0$$

$$X(x) = C\cos(\beta x) + D\sin(\beta x)$$

$$T(t) = Ae^{-k\beta^2 t}$$

$$\begin{cases} X(0)=0 \\ X(L)=0 \end{cases} \Longleftrightarrow \begin{cases} C=0 \\ D\sin(\beta L) \end{cases} \Longrightarrow \beta_n = \frac{n\pi}{L}$$

$$U_n(x,t) = D_n \sin\left(\frac{\pi n x}{L}\right) e^{-k\left(\frac{\pi n}{L}\right)^2 t}$$

$$U_N = \sum_{n=0}^{N} U_n$$

$$\phi(x) = \sum \gamma_n \sin\left(\frac{\pi n x}{L}\right)$$

$\cancel{\psi(x) = -\sum_{n=0}^{N} \alpha}$

Eigenvalue Problems

$$A(X_n) = M X_n = \lambda_n X_n$$

$$A(X_n) = \frac{d^2}{dx^2}\left(X_n(x)\right) = \lambda_n X_n(x)$$

$$X_n = \sum_{i=0}^{d} x_n^i e_i \quad (\text{Multiplication})$$

∴ usually called orthogonal polynomials

(7) Well-posedness of Problems.